Conformal Maps and p-Dirichlet Energies

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Let $\Phi\colon (M_1, g_1) \to (M_2, g_2)$ be a diffeomorphism between Riemannian manifolds and $\Phi^{\#}\colon \mathcal{D}(M_2) \to \mathcal{D}(M_1)$ the induced pull-back operator. The main theorem of this work is Theorem 4.1 which relates preservation of the p-Dirichlet energies $\varphi \mapsto \int_{M_i} |\mathrm{d}\varphi|^p \, \mathrm{d}\mu_i$ under $\Phi^{\#}$ to isometric or conformal properties of Φ. More precisely: In case $p = n$, $\Phi^{\#}$ preserves the p-Dirichlet energy if and only if Φ is conformal. In case $n \neq p \geq 2$, energy preservation is equivalent to Φ being an isometry.

Introduction

The following question was posed by Mirela Ben-Chen and was forwarded to me by Max Wardetzky:

> Let (M_1, g_1) and (M_2, g_2) be two Riemannian 2-manifolds and $\Phi\colon M_1 \to M_2$ a diffeomorphism between them. Consider the following bilinear forms on the spaces of test functions $\mathcal{D}(M_i)$, $i = 1, 2$:
>
> $$h_i\colon \mathcal{D}(M_i) \times \mathcal{D}(M_i) \to \mathbb{R}, \qquad h_i(\varphi, \psi) := \int_{M_i} \langle \mathrm{d}\varphi, \mathrm{d}\psi \rangle_{g_i} \, \mathrm{d}\mu_i$$
>
> and the linear pull-back $F := \Phi^{\#}\colon \mathcal{D}(M_2) \to \mathcal{D}(M_1)$, $\varphi \mapsto \varphi \circ \Phi$. It is well-known that $h_1(F\varphi, F\psi) = h_2(\varphi, \psi)$ holds for all φ, $\psi \in \mathcal{D}(M_2)$ if Φ is a conformal map. *Is it true that this condition is sufficient for Φ being conformal?*

In [1], Ben-Chen and her co-authors apply the bilinear forms $h_2 - F^{\#}h_1$, $(h_2 - F^{\#}h_1)|V \times V$ for finite dimensional subspaces $V \subset \mathcal{D}(M_2)$ and other derived entities as measures for non-

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conformality of Φ in the context of image comparison in computer graphic.[1] They also point out that by choosing

$$V \subset \mathcal{D}_K(M_2) = \{ \varphi \in \mathcal{D}(M_2) \mid \operatorname{supp}(\varphi) \subset K \}$$

for compact sets $K \subset M_2$, one may even localise *where* conformal distortion under Φ occurs.

By the polarization formula for symmetric bilinear forms

$$h_i(\varphi, \psi) = \tfrac{1}{2} \left(h_i(\varphi + \psi, \varphi + \psi) - h_i(\varphi, \varphi) - h_i(\psi, \psi) \right),$$

h_i can be reconstructed from $\varphi \mapsto \frac{1}{2} \int_{M_i} \langle \mathrm{d}\varphi, \mathrm{d}\varphi \rangle_{g_i} \mathrm{d}\mu_i = \frac{1}{2} |\varphi|^2_{W^{1,2}(M_i),g_i}$. The latter happens to be the Dirichlet energy of $\varphi \in \mathcal{D}(M_i)$ with respect to g_i, whereas h_i is the weak formulation of the Laplace-Beltrami operator with respect to g_i.

It is also well-known for a long time that the n-Dirichlet energy is a conformal invariant. So a natural generalisation of Ben-Chen's question in terms of n-dimensional Riemannian manifolds (M_1, g_1), (M_2, g_2) is:

> *Does*
>
> $$\tfrac{1}{n} |\Phi^{\#} \varphi|^n_{W^{1,n}(M_1),g_1} = \tfrac{1}{n} |\varphi|^n_{W^{1,n}(M_2),g_2} \quad \textit{for all } \varphi \in \mathcal{D}(M_2)$$
>
> *imply Φ to be conformal?*

As it turns out, both questions can be answered positively as will be proven in Theorem 4.1. Even more: For $n \neq p$, invariance of p-Dirichlet energy under Φ is equivalent to Φ being an isometry. Hence, the methods proposed in [1] can be used as well in higher-dimensional context as means of measuring *metric* distortion.

[1]In fact $(\mathcal{D}(M_i), h_i)$ can be extended to the Sobolev spaces $(W_0^{1,2}(M_i, g_i), h_i)$ on which h_i extends to a (non-degenerate) inner product. If M_1 and M_2 are compact, even F can be uniquely extended to $F \colon W_0^{1,2}(M_2, g_2) \to W_0^{1,2}(M_1, g_1)$. So, it is by no means necessary to test with smooth functions; any dense subspace $V \subset W_0^{1,2}(M_2, g_2)$ would do.

1 Notations and Preliminaries

We collect some notions and results on tensor bundles and on tensor bundle metrics induced by a Riemannian metric. Everthing here is supposed to be standard knowledge in the area of differential geometry, although we introduce some new notation.

Note that in the following, the ∇-operator always denotes the *covariant derivative* of a Levi-Civita connection on a Riemannian manifold. The differential of a function is denoted by d.

In order to avoid confusion, we use a raised (or lowered) "#" to mark pull-back (or push-forward) operations and reserve the raised "$*$" for adjoint operators.

For a manifold M with (Lipschitz) boundary, we denote by $M^\circ := M \setminus \partial M$ the *interior of* M. Moreover,

$$\mathcal{D}(M) := \{ \varphi \in C^\infty(M) \mid \operatorname{supp}(\varphi) \subset M^\circ \}$$

denotes the vector space of test functions. We equip it with the standard topology. That means: $\varphi_n \to \varphi$ in $\mathcal{D}(M)$ if and only if there is a compact set $K \subset M^\circ$ such that $\operatorname{supp}(\varphi_n) \subset K$ for all n and $\varphi_n|K \to \varphi|K$ in $C^\infty(K)$.

For a vector bundle $\pi\colon E \to M$, we denote by $\Gamma(M;E)$ (or $C^k\Gamma(M;E)$) the vector space of smooth (or k-times continuously differentiable) sections of π. A lowered "0" as in $\Gamma_0(M;E)$ indicates vector spaces of sections with compact support in M°.

Definition 1.1 Let M be a smooth manifold and $k \in \mathbb{Z}$. Define the *vector bundle* $\pi\colon T^kM \to M$ *of* k*-tensors* by

$$T^kM := \begin{cases} \bigotimes_{i=1}^k TM, & k > 0, \\ M \times \mathbb{R}, & k = 0, \\ \bigotimes_{i=1}^{-k} T^*M, & k < 0. \end{cases}$$

Definition 1.2 A Riemannian metric g on M induces bundle metrics

$$\langle \cdot, \cdot \rangle_{g,k} : T^kM \times_M T^kM \to T^0M$$

in the following way:

- $k = 0$: $\langle s, t \rangle_{g,0} := st$, for s, $t \in T^0_xM = \mathbb{R}$.
- $k = 1$: $\langle u, v \rangle_{g,1} := g(u,v)$, for u, $v \in T^1_xM = T_xM$.
- $k = -1$: $\langle \alpha, \beta \rangle_{g,-1} := \sum_{i,j=1}^n \alpha(u_i)\beta(u_j) \operatorname{Gram}_{\mathcal{B}}^{-1}(g)_{ij}$, for α, $\beta \in T_x^{-1}M = T_x^*M$ and with an arbitrary basis $\mathcal{B} = (u_1, \dots, u_n)$ of T_xM.
- $\langle \alpha, \beta \rangle_{g,k} := \langle \alpha_1, \beta_1 \rangle_{g,\operatorname{sgn} k} \cdots \langle \alpha_{|k|}, \beta_{|k|} \rangle_{g,\operatorname{sgn} k}$, for arbitrary simple k-tensors $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_{|k|}$, $\beta = \beta_1 \otimes \cdots \otimes \beta_{|k|}$ with α_i, $\beta_i \in T_x^{\operatorname{sgn}(k)}M$, $i = 1, \dots, |k|$.

Definition 1.3 Define $T^{(k,l)}M := T^{-k}M \otimes T^lM$ and equip it with the induced bundle metric denoted by $\langle \cdot, \cdot \rangle_{g,(k,l)}$ or denoted simply by $\langle \cdot, \cdot \rangle_g$ when the context is clear.

Definition 1.4 For $1 \leq p < \infty$ and $\alpha \in \Gamma_0(M; T^{(k,l)}M)$ define

$$\|\alpha\|_{L^p(T^{(k,l)}M,g)} := \left(\int_M \langle \alpha, \alpha \rangle_{g,(k,l)}^{\frac{p}{2}} \, \mathrm{d}\mu_g \right)^{\frac{p}{2}},$$
$$|\alpha|_{W^{1,p}(T^{(k,l)}M,g)} := \|\nabla \alpha\|_{L^p(T^{(k+1,l)}M,g)} .$$

The latter is actually a (non-complete) norm on $\Gamma_0(M; T^{(k,l)}M)$ by Poincaré's inequality. Note that $\frac{1}{p} |\alpha|_{W^{1,p}(T^{(k,l)}M,g)}^p$ is nothing but the p-Dirichlet energy of α.

Lemma 1.5 Let (M_1, g_1) and (M_2, g_2) be n-dimensional Riemannian manifolds and $\Phi\colon M_1 \to M_2$ a smooth diffeomorphism. Then one has

$$\langle \Phi^{\#}\alpha, \Phi^{\#}\beta \rangle_{g_1,(k,l)} = \Phi^{\#}(\langle \alpha, \beta \rangle_{\Phi_{\#}g_1,(k,l)}).$$

for all $x \in M_1$ and α, $\beta \in T^{(k,l)}_{\Phi(x)} M_2$.

2 Technical Lemma

This is the technical core of our result.

Lemma 2.1 Let (M, g) be a Riemannian manifold (possibly with Lipschitz boundary) with Riemannian density μ and let $A \in C^0\Gamma(M; \mathrm{End}(TM)) \cap C^1\Gamma(M^\circ; \mathrm{End}(TM^\circ))$ be a continuous symmetric (1,1)-tensor which is continuous differentiable in the interior of M. Then the following statements are equivalent:

1. $A = \mathrm{id}_{TM}$.
2. For all $p \in [2, \infty[$ and for all $\varphi, \psi \in \mathcal{D}(M)$ holds $\int_M \langle \mathrm{d}\varphi \cdot A, \mathrm{d}\psi\rangle^{\frac{p}{2}} \,\mathrm{d}\mu = \int_M \langle \mathrm{d}\varphi, \mathrm{d}\psi\rangle^{\frac{p}{2}} \,\mathrm{d}\mu$
3. There exist a $p \in [2, \infty[$ such that $\int_M \langle \mathrm{d}\varphi \cdot A, \mathrm{d}\psi\rangle^{\frac{p}{2}} \,\mathrm{d}\mu = \int_M \langle \mathrm{d}\varphi, \mathrm{d}\psi\rangle^{\frac{p}{2}} \,\mathrm{d}\mu$ holds for all $\varphi, \psi \in \mathcal{D}(M)$.

Proof. The implications 1. $\Rightarrow$ 2 and 2. $\Rightarrow$ 3 are trivial. We are going to show 3. $\Rightarrow$ 1. For $p \geq 2$, the functional $\mathcal{F}\colon \mathcal{D}(M) \to \mathbb{R}$ defined by

$$\mathcal{F}(\varphi) := \tfrac{1}{p} \int_M \Big(\langle \mathrm{d}\varphi\, A, \mathrm{d}\varphi\rangle_g^{\frac{p}{2}} - \langle \mathrm{d}\varphi, \mathrm{d}\varphi\rangle_g^{\frac{p}{2}} \Big) \,\mathrm{d}\mu$$

is Gâteaux differentiable. Its Gâteaux derivative is given by

$$\begin{aligned} D\mathcal{F}(\varphi)\,\psi &= \int_M \Big(\langle \mathrm{d}\varphi\, A, \mathrm{d}\varphi\rangle_g^{\frac{p-2}{2}} \langle \mathrm{d}\varphi\, A, \mathrm{d}\psi\rangle_g - \langle \mathrm{d}\varphi, \mathrm{d}\varphi\rangle_g^{\frac{p-2}{2}} \langle \mathrm{d}\varphi, \mathrm{d}\psi\rangle_g \Big) \,\mathrm{d}\mu \\ &= \int_M \delta\Big(\langle \mathrm{d}\varphi\, A, \mathrm{d}\varphi\rangle_g^{\frac{p-2}{2}} (\mathrm{d}\varphi\, A) - \langle \mathrm{d}\varphi, \mathrm{d}\varphi\rangle_g^{\frac{p-2}{2}} \mathrm{d}\varphi \Big) \cdot \psi \,\mathrm{d}\mu, \end{aligned}$$

where $\delta\alpha := -\operatorname{tr}_g \nabla\alpha$ for $\alpha \in \Omega^1_0(M)$ is the formal adjoint operator of d. Note that $\mathcal{F} = 0$ implies $D\mathcal{F}(\varphi)\,\psi = 0$ for all $\varphi, \psi \in \mathcal{D}(M)$. The fundamental lemma of calculus of variations leads to

$$\delta\Big(\langle \mathrm{d}\varphi\, A, \mathrm{d}\varphi\rangle_g^{\frac{p-2}{2}} (\mathrm{d}\varphi\, A) - \langle \mathrm{d}\varphi, \mathrm{d}\varphi\rangle_g^{\frac{p-2}{2}} \mathrm{d}\varphi \Big) = 0. \tag{1}$$

Observe that $\varphi \mapsto \pm\delta(\langle \mathrm{d}\varphi, \mathrm{d}\varphi\rangle_g^{\frac{p-2}{2}} \mathrm{d}\varphi)$ is the p-Laplacian (depending on sign convention). Let $X, Y \in \Gamma(M; TM)$ be smooth vector fields. Direct computations show

$$\begin{aligned} &\nabla_X\Big(\langle \mathrm{d}\varphi\, A, \mathrm{d}\varphi\rangle_g^{\frac{p-2}{2}} (\mathrm{d}\varphi\, A)\Big)(Y) \\ &\quad = \tfrac{p-2}{2} \langle \mathrm{d}\varphi\, A, \mathrm{d}\varphi\rangle_g^{\frac{p-4}{2}} \Big(2\, \langle \mathrm{d}\varphi\, A, \mathrm{Hess}(\varphi)(X, \cdot)\rangle_g + \langle \mathrm{d}\varphi\, (\nabla_X A), \mathrm{d}\varphi\rangle_g \Big) \cdot (\mathrm{d}\varphi\, A\, Y) \\ &\qquad + \langle \mathrm{d}\varphi\, A, \mathrm{d}\varphi\rangle_g^{\frac{p-2}{2}} \Big(\mathrm{Hess}(\varphi)(X, AY) + (\mathrm{d}\varphi\, (\nabla_X A)\, Y) \Big). \end{aligned}$$

Taking the trace with respect to g yields

$$\begin{aligned} &\delta\Big(\langle \mathrm{d}\varphi\, A, \mathrm{d}\varphi\rangle_g^{\frac{p-2}{2}} (\mathrm{d}\varphi\, A)\Big) \\ &\quad = -\textstyle\sum_{i=1}^n \nabla_{e_i}\Big(\langle \mathrm{d}\varphi\, A, \mathrm{d}\varphi\rangle_g^{\frac{p-2}{2}} (\mathrm{d}\varphi\, A)\Big)(e_i) \end{aligned}$$

$$= \tfrac{2-p}{2} \langle \mathrm{d}\varphi\, A, \mathrm{d}\varphi \rangle_g^{\frac{p-4}{2}} \Big(2 \left\langle \mathrm{Hess}(\varphi), (\mathrm{d}\varphi\, A) \otimes (\mathrm{d}\varphi\, A) \right\rangle_g + \langle \nabla A, (\mathrm{d}\varphi\, A) \otimes \mathrm{d}\varphi \otimes \mathrm{grad}\, \varphi \rangle_g \Big) \\ - \langle \mathrm{d}\varphi\, A, \mathrm{d}\varphi \rangle_g^{\frac{p-2}{2}} \Big(\mathrm{tr}_g(\mathrm{Hess}(\varphi)(\cdot, A \cdot)) + \mathrm{d}\varphi\ \mathrm{div}\, A \Big).$$

Now fix $x \in M^\circ$ and an orthonormal eigenbasis $e_1, \dots, e_n$ of A in T_xM with $A\, e_i = \lambda_i^2 e_i$ and $\lambda_i > 0$. Let $\varepsilon_i \in T_x^* M$ be given by $\varepsilon_i(u) := g(e_i, u)$ for all $u \in T_xM$. Then choose:

- $\varphi_j \in \mathcal{D}(M)$ with $\mathrm{d}_x \varphi_j = \varepsilon_j$ and $\mathrm{Hess}_x(\varphi_j) = g$ for $j = 1, \dots, n$.
- $\psi_j \in \mathcal{D}(M)$ with $\mathrm{d}_x \psi_j = \varepsilon_j$ and $\mathrm{Hess}_x(\psi_j) = 0$ for $j = 1, \dots, n$.

Observe $\langle \mathrm{d}\varphi_j\, A, \mathrm{d}\varphi_j \rangle_g = \langle \mathrm{d}\psi_j\, A, \mathrm{d}\psi_j \rangle_g = \lambda_j^2 \langle \varepsilon_j, \varepsilon_j \rangle_g = \lambda_j^2$. Using this and the defining properties of φ_j, ψ_j, we obtain

$$\delta\Big(\langle \mathrm{d}\varphi_j\, A, \mathrm{d}\varphi_j \rangle_g^{\frac{p-2}{2}} (\mathrm{d}\varphi_j\, A) \Big)\Big|_x \\ = \tfrac{2-p}{2} \lambda_j^{p-4} \Big(2 \langle g, \lambda_j^4 \varepsilon_j \otimes \varepsilon_j \rangle_g + \langle \nabla A, \lambda_j^2 \varepsilon_j \otimes \varepsilon_j \otimes e_j \rangle_g \Big) - \lambda_j^{p-2} \Big(\mathrm{tr}(A) + \varepsilon_j\ \mathrm{div}\, A \Big) \\ = (2-p)\lambda_j^p + \lambda_j^{p-2} \Big(\tfrac{2-p}{2} g(e_j, (\nabla_{e_j} A)\, e_j) - \mathrm{tr}(A) - g(e_j, \mathrm{div}\, A) \Big) \tag{2}$$

and analogously

$$\delta\Big(\langle \mathrm{d}\psi_j\, A, \mathrm{d}\psi_j \rangle_g^{\frac{p-2}{2}} (\mathrm{d}\psi_j\, A) \Big)\Big|_x = \lambda_j^{p-2} \Big(\tfrac{2-p}{2} g(e_j, (\nabla_{e_j} A)\, e_j) - g(e_j, \mathrm{div}\, A) \Big) \tag{3}$$

Substituting $A = \mathrm{id}_{TM}$ into (2) and (3) (hence $\lambda_j = 1$, $\nabla A = 0$, $\mathrm{div}\, A = 0$) yields

$$\delta\Big(\langle \mathrm{d}\varphi_j, \mathrm{d}\varphi_j \rangle_g^{\frac{p-2}{2}} (\mathrm{d}\varphi_j) \Big)\Big|_x = 2 - p - n, \tag{4}$$

$$\delta\Big(\langle \mathrm{d}\psi_j, \mathrm{d}\psi_j \rangle_g^{\frac{p-2}{2}} (\mathrm{d}\psi_j) \Big)\Big|_x = 0. \tag{5}$$

Combining (1), (3) and (5) shows

$$\lambda_j^{p-2} \Big(\tfrac{2-p}{2} g(e_j, (\nabla_{e_j} A)\, e_j) - g(e_j, \mathrm{div}\, A) \Big) = 0. \tag{6}$$

Inserting (6) into (2) leads to

$$(2-p)\lambda_j^p - \lambda_j^{p-2}\, \mathrm{tr}_g(A) = 2 - p - n.$$

For $C > 0$, $p \geq 2$, $n > 0$, the function $f(x) := (p-2)x^p + Cx^{p-2} + 2 - p - n$ is strictly monotonically increasing for $x \geq 0$ and fulfills $f(0) < 0$, $\lim_{x \to \infty} f(x) = \infty$. Hence, there exists exactly one solution x of $f(x) = 0$ in $]0, \infty[$ which enforces the λ_j to be all equal. This leads to

$$(2-p)\lambda_j^p - \lambda_j^{p-2} n \lambda_j^2 = 2 - p - n$$

and proves $\lambda_j^2 = 1$ for $j = 1, \dots, n$. Hence one has $A|_x = \mathrm{id}_{T_xM}$ for all $x \in M^\circ$. This holds for every $x \in M \setminus \partial M$ and for each $x \in \partial M$ by continuouity of A. ☐

3 Conformality

Here, we simply collect some well-known results from conformal geometry and some immediate consequences.

Lemma 3.1 Let (M, g) be an n-dimensional Riemannian manifold with Riemannian density μ_g and let $\lambda \in C^\infty(M; \mathbb{R})$. Consider the Riemannian metric $g_\lambda := \mathrm{e}^{2\lambda}\, g$ with Riemannian density μ_λ. Let ∇, ∇^λ be the Levi-Civita connection of g and g_λ respectively. Then one has

$$\langle \cdot, \cdot \rangle_{g_\lambda,(k,l)} = \mathrm{e}^{\lambda(l-k)} \langle \cdot, \cdot \rangle_{g,(k,l)}, \qquad \mu_\lambda = \mathrm{e}^{-\lambda n}\, \mu_g \qquad \text{and} \qquad \nabla^\lambda = \nabla.$$

Corollary 3.2 *Let $1 \le p \le \infty$, $\lambda \in C^\infty(M; \mathbb{R})$, k, l, $m \in \mathbb{N} \cup \{0\}$ and $\alpha \in \Gamma_0(M; T^{(k,l)}M)$. Then one has*

$$\|(\nabla^\lambda)^m \alpha\|_{L^p(T^{(k+m,l)}M, g_\lambda)} = \|\nabla^m \alpha\|_{L^p(T^{(k+m,l)}M, g, \mathrm{e}^{\lambda((l-k-m)p-n)}\, \mu_g)}.$$

Corollary 3.3 *Let (M, g) be a Riemannian manifold. Moreover, let $1 \le p < \infty$, k, l, $m \in \mathbb{N} \cup \{0\}$ fulfill $l - k - m + \frac{n}{p} = 0$. Then for every $\alpha \in \Gamma_0(M; T^{(k,l)}M)$, the number $\|\nabla^m \alpha\|_{L^p(T^{(k+m,l)}M, g)}$ is a conformal invariant of g.*

Corollary 3.4 *Let (M, g) be a n-dimensional Riemannian manifold. Then for every $\varphi \in \mathcal{D}(M) = \Gamma_0(M; T^{(0,0)}M)$, the n-Dirichlet energy $\|\mathrm{d}\varphi\|_{L^n(T^*M, g)}$ is a conformal invariant of g.*

Corollary 3.5 *Let (M, g) be a $2n$-dimensional Riemannian manifold. Then for every $\varphi \in \mathcal{D}(M)$, the number $\|\nabla^n \varphi\|_{L^2(T^{(n,0)}M, g)}$ is a conformal invariant of g.*

Corollary 3.6 *Let (M, g) be a 2-dimensional Riemannian manifold. Then for every $\varphi \in \mathcal{D}(M)$, the Dirichlet energy $\|\mathrm{d}\varphi\|^2_{L^2(T^*M, g)}$ is a conformal invariant of g.*

Definition 3.7 Let (M_1, g_1) and (M_2, g_2) be n-dimensional Riemannian manifolds and $\Phi\colon M_1 \to M_2$ a smooth diffeomorphism. We say Φ is a *conformal map* with respect to g_1 and g_2 if there is $\lambda \in C^\infty(M_1)$ with $\Phi^\# g_2 = \mathrm{e}^{2\lambda}\, g_1$.

4 Main Theorem

Theorem 4.1 *Let* (M_1, g_1) *and* (M_2, g_2) *be* n*-dimensional Riemannian manifolds,* $n \geq 2$*,* $\Phi\colon M_1 \to M_2$ *a smooth diffeomorphism and* $\Phi^{\#}\colon \mathcal{D}(M_2) \to \mathcal{D}(M_1)$*,* $\varphi \mapsto \varphi \circ \Phi$ *the linear pull-back map.*

1. *In case* $p = n$*,* $\Phi^{\#}\colon (\mathcal{D}(M_2), |\cdot|_{W^{1,p}(M_2,g_2)}) \to (\mathcal{D}(M_1), |\cdot|_{W^{1,p}(M_1,g_1)})$ *is an isometry of normed spaces if and only if* Φ *is a conformal map.*
2. *In case* $p \neq n$*,* $\Phi^{\#}\colon (\mathcal{D}(M_2), |\cdot|_{W^{1,p}(M_2,g_2)}) \to (\mathcal{D}(M_1), |\cdot|_{W^{1,p}(M_1,g_1)})$ *is an isometry of normed spaces if and only if* Φ *is an isometric map.*

Proof. Let $B \in \Gamma(M_2; \mathrm{End}(TM_2))$ be the symmetric, non-degenerate (1,1)-tensor $B^{-1}|_x := (T_x(\Phi^{-1}))^* T_x(\Phi^{-1})$ for $x \in M_2$, where the $(T_x(\Phi^{-1}))^*$ denotes the adjoint of

$$T_x(\Phi^{-1})\colon (T_xM_2, g_2|_x) \to (T_{\Phi^{-1}(x)}M_1, g_1|_{\Phi^{-1}(x)}).$$

Lemma 1.5 and the transformation formula for integrals imply

$$\begin{aligned}
|\Phi^{\#}\varphi|^p_{W^{1,p}(M_1,g_1)} &= \int_{M_1} \langle \mathrm{d}(\Phi^{\#}\varphi), \mathrm{d}(\Phi^{\#}\varphi)\rangle_{g_1}^{\frac{p}{2}} \,\mathrm{d}\mu_1 = \int_{M_1} \langle \Phi^{\#}\mathrm{d}\varphi, \Phi^{\#}\mathrm{d}\varphi\rangle_{g_1}^{\frac{p}{2}} \,\mathrm{d}\mu_1 \\
&= \int_{M_1} \Phi^{\#}(\langle \mathrm{d}\varphi, \mathrm{d}\varphi\rangle_{\Phi_{\#}g_1}^{\frac{p}{2}}) \,\mathrm{d}\mu_1 = \int_{M_2} \langle \mathrm{d}\varphi, \mathrm{d}\varphi\rangle_{\Phi_{\#}g_1}^{\frac{p}{2}} \frac{\mathrm{d}\Phi_{\#}\mu_1}{\mathrm{d}\mu_2} \,\mathrm{d}\mu_2 \\
&= \int_{M_2} \langle \mathrm{d}\varphi\, B, \mathrm{d}\varphi\rangle_{g_2}^{\frac{p}{2}} |\mathrm{det}_{g_2} B|^{-\frac{1}{2}} \,\mathrm{d}\mu_2 = \int_{M_2} \langle \mathrm{d}\varphi\; |\mathrm{det}_{g_2} B|^{-\frac{1}{p}} B, \mathrm{d}\varphi\rangle_{g_2}^{\frac{p}{2}} \,\mathrm{d}\mu_2,
\end{aligned}$$

where $|\mathrm{det}_{g_2} B_x|$ denotes $|\det((g_2(e_i\, B_x, e_j)_{1\leq i,j\leq n})|$ for *any* orthonormal basis $(e_1, \ldots, e_n)$ of T_xM_2 with respect to $g_2|_x$. Now, Lemma 2.1 tells us that $|\Phi^{\#}\varphi|^p_{W^{1,p}(M_1,g_1)} = |\varphi|^p_{W^{1,p}(M_2,g_2)}$ is equivalent to $|\mathrm{det}_{g_2} B|^{-\frac{1}{p}} B = \mathrm{id}_{TM_2}$. Fix $x \in M_2$ and let $\lambda_1, \ldots, \lambda_n$, $\lambda_i > 0$ be the eigenvalues of B. Note that $|\mathrm{det}_{g_2} B|^{-\frac{1}{p}} B = \mathrm{id}_{TM_2}$ is equivalent to

$$\lambda_1 = \cdots = \lambda_n = (\lambda_1 \cdots \lambda_n)^{\frac{1}{p}},$$

which means nothing but $\lambda_i = \lambda_i^{\frac{n}{p}}$. In case $n \neq p$, this implies $\lambda_i = 1$, $B = \mathrm{id}_{TM_2}$, thus Φ is an isometry of Riemannian manifolds. In case $p = n$, one can only deduce that Φ is a conformal map since all eigenvalues of B are equal. Now, Corollary 3.4 completes the proof. □

References

[1] Raif M. Rustamov, Maks Ovsjanikov, Omri Azencot, Mirela Ben-Chen, Frédéric Chazal, and Leonidas Guibas, *Map-based exploration of intrinsic shape differences and variability*, ACM Trans. Graph. **32** (2013), no. 4.