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Nonlinear Data: Theory and Algorithms

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Abstracts

Variational Convergence of Discrete Minimal Surfaces

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(joint work with Henrik Schumacher)

Consider a finite set $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3, \dots\}$ of closed embedded curves in $\mathbb{R}^m$. Among all surfaces of *prescribed* topology spanning Γ find those with least (or more precisely critical) area. Solutions of this problem are known as *minimal surfaces*—an extensively studied problem. In the 1930s, Radó and Douglas independently solved the least area problem for the special case of disk-like, immersed surfaces. To date, though, less is known about the existence of minimisers for the general case.

A natural question is how to compute minimal surfaces using finite dimensional approximations. Already Douglas followed this approach using finite differences. A more flexible option is to consider a given finite set Γ of embedded boundary curves in $\mathbb{R}^3$, followed by spanning a triangle mesh into Γ and moving the positions of interior vertices such that the overall area of the triangle mesh is minimised. Following this approach, several authors have applied Newton-like methods for finding critical points of the area functional and gradient descent (the discrete mean curvature flow) in order to produce discrete minimisers. With these tools at hand, the question remains whether the so obtained discrete minimisers converge to smooth minimal surfaces and if so in which sense?

One approach for which such convergence of discrete minimisers could be established is based on adapting Douglas' existence proof for disk-like minimal surfaces: Instead of the area of (unparameterised) surfaces, the Dirichlet energy of conformal surface parameterizations is minimised under the constraint of the so-called *three point condition*. Several authors have utilised this idea in order to compute numerical approximations of minimal surfaces via finite element analysis. However, these energy methods (which are based on minimising the Dirichlet energy instead of the area functional) face certain difficulties. E.g., in dimension greater than two, Dirichlet energy is no longer conformally invariant and minimisers of the Dirichlet energy need not minimise area.

In contrast, the Ritz method, i.e., the approach of minimising area (or volume) among simplicial manifolds, is in principle capable of treating any dimension, codimension, and topological class with a single algorithm. This is the approach we follow here. Even non-manifold examples can be treated with this method. These advantages come at a cost, though, since showing convergence of the Ritz method is hampered by several difficulties, including that (i) simplicial manifolds capture smooth boundary conditions only in an approximate sense; hence, they cannot be utilised to minimise area (or volume) in the space of surfaces with smooth prescribed boundaries, (ii) smooth minimal surfaces are known to satisfy strong regularity properties (e.g., they are analytic for sufficiently nice boundary data), leading to the question in which space and topology smooth and simplicial area minimisers ought to be compared, (iii) the general least area problem is far from being convex, and (iv) area minimisers need neither be unique nor isolated; rather, they are *sets* in general. These obstacles render the use of convex optimisation approaches and monotone operators inappropriate (if not impossible) for showing convergence of discrete (i.e., simplicial) area minimisers.

For these reasons, we suggest a different route for exploring convergence of discrete minimisers, which in particular is capable of dealing with convergence of sets. Building on variational analysis, our main result is to prove *Kuratowski convergence* of discrete area (and volume) minimisers to their smooth counterparts. While Kuratowski convergence is weaker than Hausdorff convergence in general, both notions coincide in compact metric spaces. Kuratowski convergence is related to the perhaps more familiar notion of Γ-convergence: A sequence of functionals Γ-converges if and only if their epigraphs converge in the sense of Kuratowski.

We establish Kuratowski convergence by adopting the notions of *consistency* and *stability*, following the often repeated mantra from numerical analysis that consistency and stability imply convergence. In our setting, consistency refers to the existence of sampling and reconstruction operators that take smooth manifolds to simplicial ones and vice-versa, respectively, such that the discrete and smooth area functionals stay close to one another. Stability refers to a notion of growth of sublevel sets of the smooth area functional near its (set of) minimisers. Additionally, we require the notion of *proximity*, which is motivated by finding a space in which discrete and smooth minimisers can be compared. Showing consistency, stability, and proximity is somewhat technically involved and constitutes our main technical contribution.

As a consequence of Kuratowski convergence we obtain that every cluster point of discrete area minimisers is a smooth minimal surface and every smooth minimal surface that globally minimises area is the limit of a sequence of discrete (almost) minimisers of area. For details, please refer to [1].

References

[1] H. Schumacher and M. Wardetzky, *Variational Convergence of Discrete Minimal Surfaces*, `arXiv:1605.05285`.

Reporter: Oliver Sander