Algebraic Topology on Polyhedral Surfaces from Finite Elements

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It has been known to the numerics community for some time that discretizations of smooth differential complexes such as the de Rham complex yield very stable methods for approximating solutions to partial differential equations (cf. Arnold [1]). Among the most notable such discretizations are *Whitney elements.* Given a locally finite C^∞ triangulation of a smooth manifold M, Whitney [11] defined a certain linear map W from the simplicial cochains C^q induced by this triangulation to $L^2\Lambda^q$, a chain map in the sense that $\mathrm{d}\, W = W\, \delta$, where d is the Cartan outer differential. Dozdizuk and Patodi [4][5] observed that this map together with a Riemannian metric g on a compact smooth manifold M gives rise to a positive definite inner product on simplicial cochains, and hence a discrete Hodge decomposition (using the inner product on simplicial cochains to define adjoint operators to the simplicial coboundary operators). Then under (suitable) refinement of the triangulation of M, the discrete Hodge decomposition converges to the smooth one on (M, g). Recently Wilson [12] has extended these results by a (converging) discrete wedge product on simplicial cochains and a (converging) combinatorial Hodge star operator.

Whitney elements are piecewise linear by construction. Here we report on a different development using *piecewise constant* vector fields (or one forms) on compact *polyhedral surfaces.* The function spaces corresponding to a discrete Hodge decomposition then turn out to be a mixture of *conforming* and *nonconforming* linear elements. For sequences of polyhedral surfaces whose *positions and normals* converge to the positions and normals of a compact smooth surface embedded in $\mathbb{E}^3$, we report on a convergence result for the corresponding discrete Hodge decompositions and Hodge star operators. The proof is mainly based on showing that the convergence results of Dodziuk/Patodi and Wilson remain valid if one works with variable (and converging) metrics (M, g_n), instead of a fixed one. The motivation to investigate into piecewise constant structures here is that *piecewise constant harmonic fields* come in pairs of a conforming and a nonconforming version, much alike linear models of discrete minimal surfaces [9] which also turn out to come in pairs of a conforming and a conjugate nonconforming minimal surface. Finally we remark that one finds strong similarities between the current analytic approach of discretizing function spaces (using the duality between conforming and nonconforming elements) and an algebraic approach (using the duality between primal and dual graphs), such as pursued by Desbrun et al. [3], Mercat [8], Dynnikov/Novikov [6], and others.

By a *polyhedral surface* M_h we mean the result of isometrically gluing flat Euclidean triangles along their boundaries such that the result is homeomorphic to a topological 2-manifold. As usual, h denotes the *mesh size* (a notation which goes back at least to [2]). We only consider orientable surfaces. The Euclidean structure on triangles induces a Euclidean cone structure on M_h. The triangulation gives rise to the following function spaces:

$$S_h = \{u \in C^0(M_h) \,|\, u \text{ is linear on triangles}\},$$

$$S_h^* = \{u \in L^2(M_h) \,|\, u \text{ is linear on triangles and continuous at edge midpoints}\},$$

$$\mathfrak{X}_h = \{X \text{ is tangential and constant on all individual triangles}\}.$$

Clearly, $S_h \subset S_h^*$. The space S_h is called *conforming*, and S_h^* is called *nonconforming*. Finally, $\mathfrak{X}_h$ denotes the space of *piecewise constant vector fields*. The cone metric on M_h induces a L^2-inner product on each of these spaces.

The *gradient* of a function in S_h or S_h^* is well-defined on triangles and takes values in $\mathfrak{X}_h$. Let div denote the adjoint operator to grad : $S_h \to \mathfrak{X}_h$ with respect to the L^2-inner products. Similarly, let div^* denote the adjoint operator to grad : $S_h^* \to \mathfrak{X}_h$. *Complex multiplication* J acts on $\mathfrak{X}_h$ by rotation by $\pi/2$ on each individual triangle. Set $\operatorname{curl} = -\operatorname{div} \circ \mathrm{J}$, and $\operatorname{curl}^* = -\operatorname{div}^* \circ \mathrm{J}$. It is not difficult to see that for $X \in \mathfrak{X}_h$, the terms $\operatorname{curl}^* X$ and $\operatorname{div}^* X$ are measures for the tangential and normal jumps of X across edges of M_h, respectively. If M_h is *closed* (has empty boundary), one obtains the following (mutually L^2-adjoint) chain complexes:

$$0 \longrightarrow S_h \xrightarrow{\text{grad}} \mathfrak{X}_h \xrightarrow{\operatorname{curl}^*} S_h^* \longrightarrow 0$$

$$0 \longleftarrow S_h \xleftarrow{\text{div}} \mathfrak{X}_h \xleftarrow{\mathrm{J}\,\text{grad}} S_h^* \longleftarrow 0.$$

Lemma. *The homology groups for (each of) the above chain complexes are isomorphic to the respective simplicial homology groups. This gives the following two discrete Hodge decompositions of* $\mathfrak{X}_h$*:*

$$\begin{aligned}\mathfrak{X}_h &= \operatorname{im}\operatorname{grad}_{|S_h} \oplus \operatorname{im}\mathrm{J}\operatorname{grad}_{|S_h^*} \oplus \ker\operatorname{curl}^* \cap \ker\operatorname{div}\\ &= \operatorname{im}\mathrm{J}\operatorname{grad}_{|S_h} \oplus \operatorname{im}\operatorname{grad}_{|S_h^*} \oplus \ker\operatorname{div}^* \cap \ker\operatorname{curl},\end{aligned}$$

where the second row is the J*-transformed version of the first.*

By construction, the sum is orthogonal with respect to the L^2-inner product on $\mathfrak{X}_h$. The space $\mathcal{H}(M_h;\mathbb{R}) = \ker\operatorname{curl}^* \cap \ker\operatorname{div}$ is termed *conforming harmonic*, and the space $\mathcal{H}^*(M_h;\mathbb{R}) = \ker\operatorname{div}^* \cap \ker\operatorname{curl}$ is termed *nonconforming harmonic*. The dimension of each of these spaces equals twice the genus of M_h. Note that complex multiplication J acts as a linear *isomorphism* between these two spaces. In a similar fashion to [12], one defines a *discrete Hodge star operator* on $\mathcal{H}(M_h;\mathbb{R})$ by first applying J and then L^2-projecting back to $\mathcal{H}(M_h;\mathbb{R})$,

$$\star : \mathcal{H}(M_h;\mathbb{R}) \longrightarrow \mathcal{H}(M_h;\mathbb{R}).$$

In other words, if $X \in \mathcal{H}(M_h;\mathbb{R})$, then $\star X$ is the conforming harmonic part of $\mathrm{J}(X)$. Note that $\star\star \neq -\mathrm{Id}$. However, $\star$ is still an isomorphism. There exists a similar nonconforming version.

Convergence. Let (M,g) be compact smooth surface embedded into $\mathbb{E}^3$ which inherits its metric structure from ambient space. A polyhedral surface M_h in a (small enough) tubular of M is a *normal graph* if M_h can be viewed as a section in the normal bundle of M. A sequence of normal graphs $\{M_n\}$ converges *totally normally* ([7][10]) to M if the positions of M_n converge in Hausdorff distance and the normals of M_n converge in L^∞ to those of M. Using the pullback from M_n to M, the surface M inherits a sequence of cone metrics $\{g_n\}$ coming from $\{M_n\}$.

Lemma. *If $M_n \to M$ totally normally, and X, Y are vector fields on M then*

$$\sup_{X,Y} \left\| \frac{|g_n(X,Y) - g(X,Y)|}{\|X\|_g \cdot \|Y\|_g} \right\|_\infty \longrightarrow 0.$$

Under the pullback from M_n to M, our objects are defined a.e. on M. In particular, let Π_n be the L^2-projections of smooth vector fields on M to piecewise constant fields on M associated with a totally normally converging sequence $\{M_n\}$. Then:

Theorem. *In $L^2(M)$, the components of the discrete Hodge splittings of $\Pi_n(X)$ converge to the components of the smooth Hodge splitting of X. Moreover, if $\mathfrak{h}$ is harmonic on (M,g) and $\mathfrak{h}_n$ is the conforming harmonic part of $\Pi_n(\mathfrak{h})$ then $\star_n \mathfrak{h}_n$ converges to $\star\mathfrak{h}$. Finally, $\mathcal{H}^*(M_n;\mathbb{R})$ tends to $\mathcal{H}(M_n;\mathbb{R})$, insofar as $\mathrm{J}_n\, \mathfrak{h}_n \to \star_n \mathfrak{h}_n$.*

On the one hand the proof is based showing that the convergence results proved in [4][5][12] remain true for variable and converging metrics g_n, and on the other hand on relating the Hodge splitting of Whitney elements to the Hodge splitting of piecewise constant elements. In a similar fashion one obtains convergence for the spectral decomposition of Laplacians. For details we refer to [10].

References

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