Geometric aspects of discrete elastic rods

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What this is about

Elastic rods are curve-like elastic bodies that have one dimension (length) much larger than the others (cross-section). Their elastic energy breaks down into three contributions: stretching, bending, and twisting. Stretching and bending are captured by the deformation of a space curve called the *centerline*, while twisting is captured by the rotation of a *material frame* associated to each point on the centerline. Building on the notions of framed curves, parallel transport, and holonomy, we present a smooth and a corresponding discrete theory that establishes an efficient model for simulating thin flexible rods with arbitrary cross section and undeformed configuration. To large parts, the material herein is an excerpt from [1].

Elastic energy

We describe the configuration of a smooth elastic rod by an *adapted framed curve* $\Gamma = \{\boldsymbol{\gamma}; \mathbf{t}, \mathbf{m_1}, \mathbf{m_2}\}$. Here $\boldsymbol{\gamma}(s)$ is an arc length parameterized space curve describing the rod's *centerline*; the assignment of an orthonormal material frame $\{\mathbf{t}(s), \mathbf{m_1}(s), \mathbf{m_2}(s)\}$ to each point on the centerline contains the requisite information for measuring twist. We require the material frame to be *adapted* to the centerline, *i.e.*, to satisfy $\mathbf{t}(s) = \boldsymbol{\gamma}'(s)$. As usual, we refer to $\boldsymbol{\kappa} = \mathbf{t}'$ as the centerline's *curvature (normal) vector* and to $\tau = \mathbf{m_1'} \cdot \mathbf{m_2}$ as the material frames *twist* measuring the rotation of the material around its centerline. The Kirchhoff model of elastic energy of *inextensible* (no stretching of the centerline) and *isotropic* (no preferred bending direction) elastic rods is given by

$$E = \frac{1}{2} \int_{\gamma} \alpha \boldsymbol{\kappa}^2 + \beta \tau^2 \, ds \ , \qquad (1)$$

where α and β are constants encoding bending and twisting stiffness, respectively.

Curve-angle representation & the Bishop frame

While (1) completely describes an energy model for inextensible isotropic rods, there is a more convenient description when turning to simulations—one that renders the formulation of the material frame more *explicit*. The requisite tool is provided by the *Bishop* (or parallel) frame, an adapted orthonormal frame $\{\mathbf{t}(s), \mathbf{u}(s), \mathbf{v}(s)\}$ that has zero twist uniformly, *i.e.*, $\mathbf{u}' \cdot \mathbf{v} = -\mathbf{v}' \cdot \mathbf{u} = 0$. The assignment of an adapted frame to one point on the curve uniquely pins down the Bishop frame throughout the entire curve. Every smoothly parameterized space curve with nowhere vanishing derivative carries a Bishop frame—one of several properties that sets the Bishop frame apart from the Frenet frame (which is *not* twist-free).

Denoting by θ the angle between the Bishop and the material frame in the cross section orthogonal to the centerline's tangent, *i.e.*, $\theta = \angle(\mathbf{u}, \mathbf{m_1}) = \angle(\mathbf{v}, \mathbf{m_2})$, one readily checks that the material frame's twist satisfies $\tau = \theta'$. Therefore, we can rewrite elastic energy of inextensible isotropic rods as

$$E = \frac{1}{2}\int_\gamma \alpha\boldsymbol{\kappa}^2 + \beta(\theta')^2\, ds\ . \tag{2}$$

We refer to this formulation as the *curve-angle representation*, as it previously also appeared in [3]. This representation reveals a fascinating analogy between the potential energy of elastic rods and the *kinetic* energy of Lagrange spinning tops. Indeed, by identifying the axis of the top with the direction of the rod's unit tangent, $\mathbf{t}$, and furthermore identifying the rod's arc length with the top's physical time, we find that $\int \boldsymbol{\kappa}^2 = \int (\mathbf{t}')^2$ and $\int \tau^2 = \int (\theta')^2$ measure the kinetic energy of the motion of the top's center of mass and rotation around its axis, respectively.

Holonomy & Fuller's formula

For a frame to be *parallel* along a space curve has the following interpretation. Consider the centerline's *Gauss image*, $\tilde{\boldsymbol{\gamma}}$, traced out on the unit 2-sphere, $\mathbb{S}^2$, by the unit tangent, $\mathbf{t}$. For $\{\mathbf{u}, \mathbf{v}\}$ to be parallel (twist-free) along $\boldsymbol{\gamma}$ is then equivalent for $\{\mathbf{u}, \mathbf{v}\}$ to be parallel-transported along $\tilde{\boldsymbol{\gamma}}$ in the usual sense of the Levi-Civita connection on $\mathbb{S}^2$.

Assume $\boldsymbol{\gamma}$ is a closed curve, then $\tilde{\boldsymbol{\gamma}}$ is closed as well. When parallel transporting $\{\mathbf{u}, \mathbf{v}\}$ once around $\boldsymbol{\gamma}$ (or $\tilde{\boldsymbol{\gamma}}$), the resulting final frame will usually differ from the initial one by an angle called *holonomy*, Hol. This angle is related to the so-called *writhe*. More precisely, whenever $\boldsymbol{\gamma}$ is a non self-intersecting closed space curve with (material) frame $\{\mathbf{m_1}, \mathbf{m_2}\}$, let Lk denote the (unique) *linking number* of the two curves $\{\boldsymbol{\gamma}_\pm(s)\} = \{\boldsymbol{\gamma}(s) \pm \epsilon\mathbf{m_1}(s)\}$ for some small enough $\epsilon > 0$. Then

$$Lk = Tw + Wr\ , \tag{3}$$

where $Tw = (1/2\pi)\int_\gamma \tau ds$ is the total *twist* of the material frame, while *writhe* satisfies $Wr \equiv Hol/2\pi$ modulo 1. Equation (3) is sometimes referred to as the Călugăreanu-White-Fuller formula, see, *e.g.*, [2].

Furthermore, the Gauss-Bonnet theorem implies that $Hol \equiv A$ modulo 2π, where A is the signed area enclosed by $\tilde{\boldsymbol{\gamma}}$ on $\mathbb{S}^2$.

Centerline variation

In physical simulations, in order to compute forces, we are required to express changes of elastic energy due to variations of the position (shape) of the centerline. The corresponding change of bending energy, $\int_\gamma \boldsymbol{\kappa}^2 ds$, is straightforward to calculate, while computing the change of twisting energy, $\int_\gamma \tau^2 ds$, is slightly more involved since it requires the computation of the change of holonomy (or writhe). If $\boldsymbol{\gamma}$ is a closed curve, then it follows from Gauss-Bonnet that the change in holonomy, δHol, with respect to varying the centerline's *tangent* (the position of $\tilde{\boldsymbol{\gamma}}$ on

$\mathbb{S}^2$) by $\delta\mathbf{t}$ is given by

$$(4) \qquad \delta Hol = \delta A = -\int_\gamma \delta\mathbf{t}\cdot(\mathbf{t}\times\mathbf{t}')\,ds \quad \text{and hence} \quad \delta\theta' = \delta\mathbf{t}\cdot(\kappa\mathbf{b})\ ,$$

where $\kappa\mathbf{b} = \mathbf{t}\times\mathbf{t}'$ is the centerline's curvature binormal vector. For closed cuves, (4) is the infinitesimal version of Fuller's calculation for the difference between the writhe of two space curves, see [2]. From (4) we obtain that the L^2-gradient of Hol with respect to variations of positions (not tangents) is $(\kappa\mathbf{b})'$.

The discrete picture

We represent a discrete rod's centerline as a piecewise straight polygonal space curve, and we associate discrete adapted orthonormal frames with *edges* of this curve. Along each edge we assume these frames to be constant. For each pair $(\mathbf{e}_{i-1}, \mathbf{e}_i)$ of consecutive edges, *discrete parallel transport* from one edge to the next is given by rotating by the angle $\angle(\mathbf{e}_{i-1}, \mathbf{e}_i)$ about the normal to the plane spanned by $\mathbf{e}_{i-1}$ and $\mathbf{e}_i$. This gives rise to discrete Bishop (parallel) frames. Accordingly, we obtain a discrete notion of holonomy (or writhe) for closed polygonal curves.

We require elastic energy and hence a discrete notion of curvature and twist. As in the smooth case, twist is nothing but the change of the angle between the Bishop and the material frame at each edge of the polygonal curve, $\boldsymbol{\gamma}$.

Consider once more the Gauss image, $\tilde{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$, on $\mathbb{S}^2$. The vertices of $\tilde{\boldsymbol{\gamma}}$ correspond to unit tangents, $\mathbf{t}_i$, along the edges of $\boldsymbol{\gamma}$, while the edges of $\tilde{\boldsymbol{\gamma}}$ are arcs of great circles. As in the smooth case before, Gauss-Bonnet tells us that discrete holonomy is related to the signed area enclosed by $\tilde{\boldsymbol{\gamma}}$. To obtain forces, it therefore suffices to study variations of this area with respect to variations of the vertices of $\tilde{\boldsymbol{\gamma}}$. To this end, consider an arc of a great circle of length $\phi_i = \angle(\mathbf{t}_{i-1}, \mathbf{t}_i) < \pi$ between $\mathbf{t}_{i-1}$ and $\mathbf{t}_i$ and consider respective variations by $\delta\mathbf{t}_{i-1}$ and $\delta\mathbf{t}_i$ on $\mathbb{S}^2$. Consider further the area swept out by the geodesics that connect the two varying endpoints. It follows (for example by considering Jacobi fields) that this area (and therefore discrete holonomy) satisfies

$$(5) \qquad \delta Hol = \delta A = -\frac{\delta\mathbf{t}_{i-1} + \delta\mathbf{t}_i}{2}\cdot\left(2\tan\frac{\phi_i}{2}\mathbf{b}_i\right) \quad \text{with} \quad \mathbf{b}_i = \frac{\mathbf{t}_{i-1}\times\mathbf{t}_i}{|\mathbf{t}_{i-1}\times\mathbf{t}_i|},$$

which is the discrete analogue of (4). By *postulating* in the discrete case relation (4) between the gradient of holonomy and curvature, we may define discrete curvatures at the vertices of $\boldsymbol{\gamma}$ by $\kappa_i = 2\tan(\phi_i/2)$, where ϕ_i is the angle between the edges incident to a particular vertex.

For additional material, including anisotropic rods and simulation results, see [1].

References

[1] M. Bergou, M. Wardetzky, S. Robinson, B. Audoly, and E. Grinspun. Discrete elastic rods. In *ACM Transactions on Graphics (ACM SIGGRAPH)*, 27(3), 2008.

[2] F. B. Fuller. Decomposition of the linking number of a closed ribbon: A problem from molecular biology. *PNAS*, 75(8):3557–3561, 1978.

[3] J. Langer and D. A. Singer. Lagrangian Aspects of the Kirchhoff Elastic Rod. *SIAM Review*, 38(4):605–618, 1996.