**Persistence simplification of discrete Morse functions on surfaces (Oberwolfach report)**

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## 1. Introduction

We apply the concept of persistent homology [1] to Forman's discrete Morse theory [2] on regular 2-manifold CW complexes and solve the problem of minimizing the number of critical points among all functions within a prescribed distance $\delta$ from a given input function. Our result achieves a lower bound on the number of critical points and improves on previous work [3] by a factor of two.

## 2. Discrete Morse theory

Let $\mathcal{K}$ be a finite CW complex and $K$ the set of cells of $\mathcal{K}$. The cell $\sigma$ is a *face* of $\tau$, denoted by $\sigma < \tau$, if $\sigma$ is in the boundary of $\tau$. *Facets* are faces of codimension 1. If the attaching maps are homeomorphisms, $\mathcal{K}$ is called a *regular* complex. A *combinatorial surface* is a regular CW complex whose underlying space is a 2-manifold.

Discrete vector fields are one of the central concepts of discrete Morse theory. They are a purely combinatorial analogon of classical vector fields.

**Definition** (Discrete vector field)**.** *A* discrete vector field $V$ *on a regular CW complex* $\mathcal{K}$ *is a set of pairs of cells* $(\sigma, \tau) \in K \times K$*, with* $\sigma$ *a facet of* $\tau$*, such that each cell of* $K$ *is contained in at most one pair of* $V$*.*

**Definition** ($V$-path)**.** *Let* $V$ *be a discrete vector field. A* $V$*-path* $\Gamma$ *from a cell* $\sigma_0$ *to a cell* $\sigma_r$ *is a sequence* $\sigma_0\tau_0\sigma_1 \dots \tau_{r-1}\sigma_r$ *of cells such that for every* $0 \le i \le r-1$*:*

$$\sigma_i \text{ is a facet of } \tau_i \quad \text{and} \quad (\sigma_i, \tau_i) \in V,$$
$$\sigma_{i+1} \text{ is a facet of } \tau_i \quad \text{and} \quad (\sigma_{i+1}, \tau_i) \notin V.$$

*A* $V$*-path is a* nontrivial closed path *if* $\sigma_0 = \sigma_r$ *and* $r > 0$.

**Definition** (Discrete gradient vector field)**.** *A* gradient vector field *is a discrete vector field* $V$ *that does not admit any nontrivial closed* $V$*-paths.*

**Definition** (Critical cell)**.** *A cell* $\sigma$ *is a* critical cell *with respect to a discrete gradient vector field* $V$ *if* $\sigma$ *is not contained in any pair of* $V$*. A cell that is not critical is a* regular cell*.*

The main technique for reducing the number of critical points is that of *reversing* a gradient vector field $V$ along a $V$-path between two critical cells $\tau$ and $\sigma$:

**Theorem** ([2], Theorem 11.1)**.** *Let* $\sigma$ *and* $\tau$ *be critical cells of a gradient vector field* $V$ *with a unique* $V$*-path* $\Gamma$ *from* $\partial\tau$ *to* $\sigma$*. Then there is a gradient vector field* $\widetilde{V}$ *obtained by reversing* $V$ *along the path* $\Gamma$*. The critial cells of* $\widetilde{V}$ *are exactly the critical cells of* $V$ *other than* $\{\sigma, \tau\}$*. In particular,* $V = \widetilde{V}$ *except along the path* $\Gamma$*.*

As in smooth Morse theory, a discrete gradient vector field can be understood as the gradient of some function in the following sense:

**Definition** ((Pseudo-)Morse function)**.** *A* discrete Morse function *is a function $f : K \to \mathbb{R}$ on the cells of a regular CW complex $\mathcal{K}$ if there is a gradient vector field $V$ such that for all pairs of cells we have*

$$\sigma \text{ is a facet of } \tau \Rightarrow \begin{cases} f(\sigma) < f(\tau) & \text{if } (\sigma, \tau) \notin V, \\ f(\sigma) \geq f(\tau) & \text{if } (\sigma, \tau) \in V. \end{cases}$$

*For a* discrete pseudo-Morse function*, the strict inequality is replaced by a weak one, i.e., $f(\sigma) \leq f(\tau)$ if $(\sigma, \tau) \notin V$. In either case, we call $V$* consistent with $f$*.*

**Definition** (Induced partial order)**.** *The partial order $\prec_V$* induced *by a discrete gradient vector field $V$ is the transitive relation generated by*

$$\sigma \text{ is a facet of } \tau \Rightarrow \begin{cases} \sigma \prec_V \tau & \text{if } (\sigma, \tau) \notin V, \\ \sigma \succ_V \tau & \text{if } (\sigma, \tau) \in V. \end{cases}$$

*For any pseudo-Morse function $g$ consistent with $V$ and any pair of cells $(\phi, \rho)$, $\phi \prec_V \rho$ implies $g(\phi) \leq g(\rho)$.*

## 3. Persistent Morse homology

Homological persistence [1] is used to investigate the change of the homology groups in a sequence of nested topological spaces. We study nested subcomplexes of a given CW complex.

**Definition** (Level subcomplex)**.** *Let $f$ be a discrete Morse function on a regular CW complex $\mathcal{K}$. For a cell $\sigma \in K$, the* level subcomplex *is the subcomplex of $\mathcal{K}$ consisting of all cells $\rho$ with $f(\rho) \leq f(\sigma)$ together with their faces:*

$$\mathcal{K}(\sigma) := \bigcup_{\substack{\rho \in K \\ f(\rho) \leq f(\sigma)}} \bigcup_{\substack{\phi \in K \\ \phi \leq \rho}} \phi \,.$$

For $\mathcal{K}(\phi) \subset \mathcal{K}(\rho)$, let $i_*^{\phi,\rho} : H_*(\mathcal{K}(\phi)) \to H_*(\mathcal{K}(\rho))$ denote the homomorphism induced by inclusion. Let $\sigma$ and $\tau$ be critical cells of dimension $d$ and $(d + 1)$, respectively, such that $f(\sigma) < f(\tau)$. The *predecessor* of $\sigma$ is the cell $\sigma_-$ with the largest $f$-value such that $f(\sigma_-) < f(\sigma)$, and similarly for $\tau_-$. Now consider the sequence

$$H_d(\mathcal{K}(\sigma_-)) \to H_d(\mathcal{K}(\sigma)) \to H_d(\mathcal{K}(\tau_-)) \to H_d(\mathcal{K}(\tau))$$

induced by inclusion.

**Definition** (Birth, death, persistence pair)**.** *Let $f$ be an injective Morse function on a regular CW complex. We say that a class $h \in H_*(\mathcal{K}(\sigma))$ is* born at *(or* created by*) $\sigma$ if*

$$h \notin \operatorname{im}(i_*^{\sigma_-,\sigma}).$$

*Moreover, we say that a class $h \in H_*(\mathcal{K}(\sigma))$ that is born at $\sigma$* dies entering *(or* gets merged by*) $\tau$ if*

$$i_d^{\sigma,\tau}(h) \in \mathrm{im}(i_*^{\sigma_-,\tau}) \quad \textit{but} \quad i_d^{\sigma,\tau_-}(h) \notin \mathrm{im}(i_*^{\sigma_-,\tau_-}).$$

*If there exists a class $h$ that is born at $\sigma$ and dies entering $\tau$, then $(\sigma, \tau)$ is a* persistence pair*. The difference $f(\tau) - f(\sigma)$ is called the* persistence *of $(\sigma, \tau)$.*

To uniquely define persistence pairs for a *pseudo-Morse* function $f$ consistent with some gradient vector field $V$, we require a total order on the cells. This can be achieved by extending the partial order $\prec_V$ to a total order, which allows us to speak about persistence pairs of $(f, V)$.

## 4. Topological simplification of functions

From now on, let $f$ be a pseudo-Morse function consistent with a discrete gradient vector field $V$ on a combinatorial surface $\mathcal{K}$. From the stability theorem for persistence diagrams [4], we can deduce the following lower bound on the number of persistence pairs, and therefore on the number of critical points:

**Lemma.** *For a pseudo-Morse function $f_\delta$ with $\|f_\delta - f\|_\infty < \delta$ and consistent with a gradient vector field $V_\delta$, the number of persistence pairs of $(f_\delta, V_\delta)$ is bounded from below by the number of persistence pairs of $f$ with persistence $\geq 2\delta$.*

We are interested in functions that achieve this lower bound:

**Definition** (Perfect $\delta$-simplification)**.** *A* perfect $\delta$-simplification *of $(f, V)$ is a pseudo-Morse function $f_\delta$ consistent with a gradient vector field $V_\delta$, such that $\|f_\delta - f\|_\infty < \delta$ and the number of persistence pairs of $(f_\delta, V_\delta)$ is equal to the number of persistence pairs of $f$ with persistence $\geq 2\delta$.*

Our main result states that a perfect $\delta$-simplification always exists for a discrete pseudo-Morse function on a combinatorial surface.

**Theorem.** *Let $f$ be a discrete pseudo-Morse function on a combinatorial surface. Then there exists a perfect $\delta$-simplification of $f$.*

The proof of this theorem is constructive. An analogous statement is not true in higher dimensions or for non-manifold complexes.

## References


[1] H. Edelsbrunner, D. Letscher, and A. Zomorodian. Topological persistence and simplification. *Discrete and Computational Geometry*, 28(4):511–533, 2002.

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[3] H. Edelsbrunner, D. Morozov, and V. Pascucci. Persistence-sensitive simplification of functions on 2-manifolds. In *SCG '06: Proceedings of the 22nd ACM Symposium on Computational Geometry*, pages 127–134. ACM, 2006.

[4] D. Cohen-Steiner, H. Edelsbrunner, and J. Harer. Stability of persistence diagrams. *Discrete and Computational Geometry*, 37(1):103–120, 2007.