Towards a curvature theory for general quad meshes

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Motivation from integrable geometry

Surfaces of constant (mean or Gauss) curvature are often expressed in special parametrizations (e.g., conformal curvature line or asymptotic line), where their integrability equations reduce to completely integrable ("soliton") PDEs from theoretical physics. For each of these specially parametrized constant curvature surfaces, there exists a one parameter *associated family* that stays within the surface class (e.g., minimal surfaces stay minimal), but the parametrization may change.

Discrete partial finite difference analogues for many of these integrable PDEs have been discovered, where discrete integrability is encoded by a certain closing condition around a 3D cube [5]. In this way algebraic constructions for discrete analogues of different classes of surfaces (reviewed in [3]), such as minimal surfaces, surfaces of constant mean curvature, and constant negative Gauss curvature, have been described, together with their one parameter associated families. However, a general notion of curvatures for these surfaces that is constant for the appropriate algebraic constructions has been lacking.

Investigating the geometry of these algebraic families leads us to the following curvature theory, which not only retrieves the thoroughly investigated curvature definitions in the case of planar quads [8, 4], but extends to the general setting of nonplanar quad meshes.

Quad meshes as discrete parametrized surfaces

A natural discrete analogue of a smooth parametrized surface patch is given by a *quad mesh patch*, a map $f : D \subset \mathbb{Z}^2 \to \mathbb{R}^3$ with nonvanishing straight edges. Notice, in particular, that the quadrilateral faces in $\mathbb{R}^3$ may be *nonplanar*. Piecing together such patches defines a *quad mesh* with more general combinatorics and is understood as an atlas for a surface. To every vertex of a quad mesh we associate a unit "normal" vector and define the corresponding map, written per patch as $n : D \subset \mathbb{Z}^2 \to \mathbb{S}^2$, as the *discrete Gauss map*. An arbitrary Gauss map will not be "normal" to its quad mesh, so we only allow for those Gauss maps that satisfy a certain constraint along each edge. The resulting class of *edge-constraint quad meshes* is our focus.[1]

Definition. *A quad mesh f with Gauss map n is called* edge-constraint *if for every edge of f in $\mathbb{R}^3$ the average of the normals at its end points is perpendicular to f, i.e., for $i = 1, 2$ we have $(f_i - f) \cdot \frac{1}{2}(n_i + n) = 0$.*

[1] We make use of *shift notation* to describe quantities per quad: $f := f(k, \ell), f_1 := f(k+1, \ell)$, and $f_2 := f(k, \ell+1)$, so $f_{12} := f(k+1, \ell+1)$; n, n_1, n_2, and n_{12} are defined similarly.

Curvatures from offsets. For an edge-constraint quad mesh f with Gauss map n, the *offset family* is given by adding multiples of the Gauss map to each vertex, i.e., $f^t := f + tn$. Notice that every mesh of the offset family is edge-constraint with the same Gauss map. In the smooth setting, corresponding tangent planes between a surface, its normal offsets, and common Gauss map are parallel. This allows one to compare areas and derive curvatures.

In order to mimic this definition of curvatures in the discrete case, let $Q_n = (n, n_1, n_{12}, n_2)$ be a Gauss map quad with corresponding quad $Q_f = (f, f_1, f_{12}, f_2)$ and offset quads Q_{f^t}. Define the *discrete Gauss map partial derivatives* as the midpoint connectors of Q_n, i.e., $n_x := \frac{1}{2}(n_{12} + n_1) - \frac{1}{2}(n_2 + n)$ and $n_y := \frac{1}{2}(n_{12} + n_2) - \frac{1}{2}(n_1 + n)$. We define the *common quad tangent plane* between Q_n, Q_f, and Q_{f^t} as the plane spanned by n_x, n_y; and we call $N := \frac{n_x \times n_y}{\|n_x \times n_y\|}$ the *projection direction.*[2] The midpoint connectors of Q_f and Q_{f^t} do not lie in this common tangent plane, so we project them to the plane spanned by n_x and n_y. The partial derivatives f_x, f_y and f^t_x, f^t_y are each defined as the projection (induced by N) of the corresponding midpoint connectors.

The curvature theory for edge-constraint quad meshes is now built per quad and mimics the smooth setting.

Definition. *The* mixed area form *per quad of two quad meshes g, h sharing a Gauss map n is given by $A(g, h) := \frac{1}{2}(\det(g_x, h_y, N) + \det(h_x, g_y, N))$.*

Note that when corresponding quads of g, h and n are in fact planar, lying in parallel planes, and $h = g$ then the quantity $A(g, g)$ coincides with the usual area of a quad.

Lemma. *The area of an offset quad satisfies the* Steiner formula*: $A(f^t, f^t) = A(f, f) + 2tA(f, n) + t^2 A(n, n)$.*

As in the smooth setting, factoring out $A(f, f)$ defines the mean and Gauss curvature.

Definition. *The mean and Gauss curvature per quad of an edge-constraint quad mesh are given by $\mathcal{H} := \frac{A(f,n)}{A(f,f)}$ and $\mathcal{K} := \frac{A(n,n)}{A(f,f)}$, respectively.*

Fundamental forms, shape operator, and principal curvatures. Fundamental forms of a parametrized surface can be written in terms of the partial derivatives in the tangent plane at each point. The same formulas define these objects per common quad tangent plane of an edge-constraint quad mesh.

Definition. *The* fundamental forms *are defined as* $\mathrm{I} := \begin{pmatrix} f_x \cdot f_x & f_x \cdot f_y \\ f_y \cdot f_x & f_y \cdot f_y \end{pmatrix}$, $\mathrm{II} := \begin{pmatrix} f_x \cdot n_x & f_x \cdot n_y \\ f_y \cdot n_x & f_y \cdot n_y \end{pmatrix}$, *and* $\mathrm{III} := \begin{pmatrix} n_x \cdot n_x & n_x \cdot n_y \\ n_y \cdot n_x & n_y \cdot n_y \end{pmatrix}$. *The shape operator is given by* $\mathrm{S} := \mathrm{I}^{-1}\mathrm{II}$.

[2]To include the cases when n_x and n_y are parallel (corresponding to *developable*, i.e., vanishing Gauss curvature, surfaces), we in fact define a *family of projection directions* $U := \{N \in \mathbb{S}^2 | \ N \perp \operatorname{span}\{n_x, n_y\}\}$. The mean and Gauss curvatures are invariant to the choice of $N \in U$.

Observe that the Gauss map being normal to the surface guarantees the existence of principal curvatures and curvature lines.

Lemma. *The edge-constraint implies that the second fundamental form is symmetric ($f_x \cdot n_y = n_x \cdot f_y$), so the shape operator is diagonalizable.*

Definition. *The real eigenvalues k_1, k_2 of the shape operator are the* principal curvatures *per quad. The corresponding eigenvectors yield curvature directions in each quad tangent plane.*

The expected relationships hold between the principal curvatures, fundamental forms, and the mean and Gauss curvatures defined via the Steiner formula.

Lemma. *The following are true in the smooth and discrete case:*
1. $\mathcal{K} = k_1 k_2 = \det \mathrm{II} / \det \mathrm{I}$, *2.* $\mathcal{H} = \frac{1}{2}(k_1 + k_2)$,
3. $\mathrm{III} - 2\mathcal{H}\mathrm{II} + \mathcal{K}\mathrm{I} = 0$, *and* *4.* $A(f, f)^2 = \det \mathrm{I}$.

Constant curvature quad meshes

It turns out that many integrable geometries are edge-constraint quad meshes of the appropriate curvature; an example of non integrable geometry is recovered, too. For more details see [6].

Theorem. *The following previously defined algebraic quad meshes are edge-constraint of the appropriate* constant *curvature:*
1. *Discrete minimal* [2] *and their associated families,*
2. *Discrete cmc* [3] *and their associated families,*
3. *Discrete constant negative Gauß curvature* [1] *and their associated families,*

Theorem. *Discrete developable quad meshes built from planar strips* [7] *can be extended to edge-constraint quad meshes with vanishing Gauss curvature.*

References

[1] A. I. Bobenko and U. Pinkall. Discrete surfaces with constant negative Gaussian curvature and the Hirota equation. *Journal of Differential Geometry*, 43:527–611, 1996.
[2] A. I. Bobenko and U. Pinkall. Discrete isothermic surfaces. *Journal für die reine und angewandte Mathematik*, pages 187–208, 1996.
[3] A. I. Bobenko and U. Pinkall. Discretization of Surfaces and Integrable Systems. In A. I. Bobenko and R. Seiler, editors, *Discrete integrable geometry and physics*, pages 3–58. Oxford University Press, 1999.
[4] A. I. Bobenko, H. Pottmann, and J. Wallner. A curvature theory for discrete surfaces based on mesh parallelity. *Mathematische Annalen*, 348(1):1–24, 2010.
[5] A. I. Bobenko and Y. B. Suris. *Discrete Differential Geometry: Integrable Structure*, volume 98 of *Graduate Studies in Mathematics*. American Mathematical Society, 2008.
[6] T. Hoffmann, A. O. Sageman-Furnas, and M. Wardetzky. A discrete parametrized surface theory in $\mathbb{R}^3$. *arXiv preprint:1412.7293.*
[7] Y. Liu, H. Pottmann, J. Wallner, Y.-L. Yang, and W. Wang. Geometric modeling with conical meshes and developable surfaces. *ACM Transactions on Graphics*, 25(3):681–689, 2006.
[8] W. K. Schief. On a maximum principle for minimal surfaces and their integrable discrete counterparts. *Journal of Geometry and Physics*, 56(9):1484–1495, 2006.