A 2×2 Lax representation, associated family, and Bäcklund transformation for circular K-nets*

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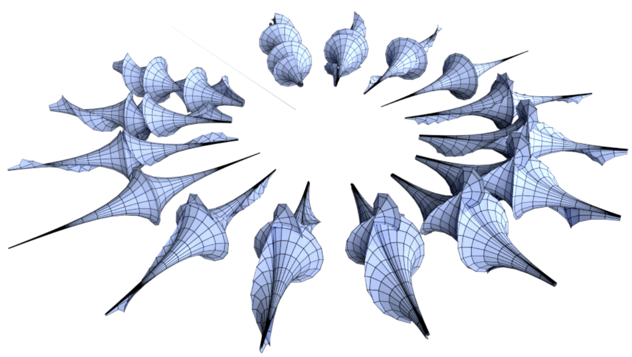

Figure 1: Bäcklund transformations of the discrete curvature line Pseudosphere aligned by angular parameter.

Abstract

We present a 2×2 Lax representation for discrete circular nets of constant negative Gauß curvature. It is tightly linked to the 4D consistency of the Lax representation of discrete K-nets (in asymptotic line parametrization). The description gives rise to Bäcklund transformations and an associated family. All the members of that family – although no longer circular – can be shown to have constant Gauß curvature as well. Explicit solutions for the Bäcklund transformations of the vacuum (in particular Dini's surfaces and breather solutions) and their respective

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associated families are given.

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1 Introduction

Smooth surfaces of constant negative curvature and their transformations are a classical topic of differential geometry (for a modern treatment see, e.g., the book by Rogers and Schief [**?**]).

Discrete analogues of surfaces of constant negative Gauß curvature in *asymptotic parametrization* (now known as *K-nets*) and their Bäcklund transformations were originally defined by Wunderlich [**?**] and Sauer [**?**] in the early 1950s. In 1996 Bobenko and Pinkall [**?**] showed that these geometrically defined K-nets are equivalent to ones arising algebraically from a discrete moving frame 2x2 Lax representation of the well known discrete Hirota equation [**?**]. This algebraic viewpoint highlights the interrelationship between a discrete net, its associated family (generated by the spectral parameter of the Lax representation, which corresponds to reparametrization of the asymptotic lines), and its Bäcklund transformations (arising from the 3D consistency of the underlying discrete evolution equation).

In the smooth setting, surface reparametrization is a simple change of variables and does not affect the underlying geometry. However, understanding surface reparametrization in the discrete setting is a much more delicate issue. In particular, discrete analogues of constant negative Gauß curvature surfaces in *curvature line parametrizations* have been defined and studied by restricting a notion of discrete curvature line parametrization (called *C-nets* since each quad is concircular) with its corresponding definition of Gauß curvature [**?**, **?**, **?**, **?**]. We call such objects *circular K-nets* or *cK-nets* and will be our main focus. Recently, a curvature theory has been introduced for a more general class of nets (so-called *edge-constraint* nets) that furnishes both asymptotic K-nets and cK-nets with constant negative Gauß curvature [**?**].

In [**?**] Schief gave a Lax representation for circular K-nets in terms of 3×3 matrices in the framework of a special reduction of C-nets and showed how circular 3D compatibility cubes give rise to Bäcklund transformations. However, this Bäcklund transformation corresponds to a double Bäcklund transformation in the smooth setting and the relationship between cK-nets and asymptotic K-nets remained unclear.

In what follows we show that cK-nets, discrete curvature line nets of constant negative Gauß curvature, exhibit a 2×2 Lax pair, associated family, and Bäcklund transformations that naturally arise from their construction from K-nets; each cK-net Lax matrix is the product of two K-net matrices, with opposite angular parameters. In other words, each edge of a cK-net is a diagonal of a rhombic (equal edge length) K-net quad. This is reasonable and expected since curvature coordinates are the sum and difference of asymptotic ones. However, we wish to highlight three important subtleties that arise:

1. A generic cK-net has quads lying on circles of varying radii; even though each edge factors into a diagonal of a rhombic K-net quad, the four corresponding rhombi do not share a central vertex (see Figure 5).

2. The associated family of cK-nets yields nets in more general parametrization (since the spectral parameter corresponds to reparametrization of the asymptotic lines), but as shown in Theorem 6 they are all constant negative Gauß curvature edge-constraint nets.

3. As shown in Figure 8, the 3D compatibility cube corresponding to the Bäcklund transformation of cK-nets is unusual since the equation for its sides is not the same as that of its top and bottom. However, double Bäcklund transformations with negative angular parameters do form a usual 3D consistent cube with circular faces. These double Bäcklund transformations also accept complex angular parameters, yielding non-factorizable breather surfaces, as shown in Figure 10.

The paper is organized as follows: after introducing some preliminaries, we recapitulate many facts about asymptotic K-nets. Then we briefly review the recently introduced theory of edge-constraint nets and their curvatures. The main results are in Section 2. In Section 2.2 we define Lax matrices and prove that they give rise to edge-constraint nets. In 2.3 we see that these are in fact Lax matrices for cK-nets (and their associated families) and that every cK-net arises in this way. The Bäcklund transformation for cK-nets is given in Section 2.4. Finally, in Section 2.5 we present closed form equations for some Bäcklund transformations of the straight line, yielding discrete analogues of, e.g., Dini's surfaces, Kuen's surface, and breather surfaces, together with their respective associated families.

1.1 Preliminaries and notation

We consider a discrete analogue of parametrized surfaces in $\mathbb{R}^3$ known as quad nets.

Definition 1. *A* quad graph G *is a strongly regular polytopal cell decomposition of a regular surface with all faces being quadrilaterals. A* quad net *is an immersion of a quad graph into* $\mathbb{R}^3$*.*

For simplicity we assume G to be $\mathbb{Z}^2$ in the following sections, though all results generalize to *edge-bipartite* quad graphs.[1] Furthermore, we associate a unit vector (or *normal*) to each vertex of a quad net, equipping it with a discrete Gauß map $n : G \to \mathbb{S}^2$. This is further explained in Section 1.3.

To distinguish arbitrary vertices of a quad net (or its Gauß map) we use shift notation. For $f : \mathbb{Z}^2 \to \mathbb{R}^3$, f denotes the map at a vertex (k, ℓ) and subsequent

[1] For more general edge-bipartite graphs, vertices with valence greater than four might not have a continuous limit in the classical sense. For example, if the quad net is a discrete constant negative Gauß curvature surface parametrized by curvature lines, then these points are something like "Lorentz umbilics" [?].

subindices stand for shifts in the corresponding *lattice directions*: $f = f(k, \ell)$, $f_1 := f(k+1, \ell)$, $f_2 := f(k, \ell+1)$, and $f_{12} = f(k+1, \ell+1)$.

Discrete integrable surface theory has well established analogues of asymptotic (A-net) and curvature line (C-net) parametrizations [**?**, **?**].

Definition 2. *An* A-net *is a quad net where each vertex star lies in a plane.*

Definition 3. *A* C-net *is a quad net where each face is inscribed in a circle.*

1.2 (asymptotic) K-nets

The theory of K-nets – discretizations of surfaces of constant negative Gauß curvature in *asymptotic* line parametrization – is well established (see, for example, [**?**, **?**, **?**, **?**, **?**, **?**, **?**]). We briefly review three of its aspects: (i) the *geometric* K-net definition, (ii) the *algebraic* K-net definition and its equivalence to the geometric one, and (iii) the relationship between their Bäcklund transformations and multidimensional consistency. In Section 2 we investigate these aspects for discretizations of surfaces of constant negative Gauß curvature in *curvature* line parametrization, which we call *cK-nets*.

K-nets were originally investigated by Sauer [**?**] and Wunderlich [**?**] both of whom used a geometric definition similar to the following.

Definition 4. *A* geometric K-net $f : \mathbb{Z}^2 \to \mathbb{R}^3$ *is an A-net such that*

i. every quad is a skew parallelogram*, i.e.,* f, f_1, f_{12}, f_2 *satisfies* $\|f_1 - f\| = \|f_{12} - f_2\|$ *and* $\|f_2 - f\| = \|f_{12} - f_1\|$*,*

ii. there exists a Gauß map $n : \mathbb{Z}^2 \to \mathbb{S}^2$ *determining an orientation at each planar vertex star, such that all interior angles of each skew parallelogram are nonnegative, and*

iii. The signed dihedral angle $\delta_{(i)}$ *between neighboring normals along each edge is measured oppositely for the two lattice directions, i.e.,* $\delta_{(1)} = \sphericalangle(n, n_1) \in [0, \pi]$ *and* $\delta_{(2)} = \sphericalangle(n, n_2) \in [-\pi, 0]$.

Remark 1. *We call a geometric K-net* rhombic *if every skew parallelogram is in fact a rhombus, so all edges of the net are of the same length.*

The notation for dihedral angles indicates a dependence on only one lattice direction. This is highlighted by the following result, which is an immediate consequence of the 180° symmetry of a skew parallelogram.

Lemma 1. *Geometric K-net dihedral angles* $\delta_{(i)}$ *are functions that depend only on shifts in their parenthetically subscripted index, i.e.,* $\delta_{(1)} = \sphericalangle(n, n_1) = \sphericalangle(n_2, n_{12})$ *and* $\delta_{(2)} = \sphericalangle(n, n_2) = \sphericalangle(n_1, n_{12})$*.*

Remark 2. *Each Gauß map quad of a geometric K-net is also a skew parallelogram.*

We do not consider geometric K-nets that lie entirely in a plane (where each skew parallelogram is in fact a planar parallelogram). A natural measure of skewness (or non-planarity) of a parallelogram is given by its *folding parameter.*

Lemma 2. *For a geometric K-net skew parallelogram we have* $\frac{\sin\delta_{(1)}}{\|f_1-f\|} = -\frac{\sin\delta_{(2)}}{\|f_2-f\|}$*, defining its* folding parameter*. The folding parameter of a geometric K-net is constant.*

Proof. Consider the tetrahedron spanned by the vertices of the skew parallelogram f, f_1, f_{12}, f_2 then n, n_1, n_{12}, n_2 is not only a geometric K-net Gauß map, but also a set of face normals on the four triangular faces of this tetrahedron. By 180° symmetry of the skew parallelogram, the product of the areas of the two triangular faces of the tetrahedron incident to each edge of the parallelogram is constant, which we denote $c > 0 \in \mathbb{R}$. The tetrahedron law of cosines gives the volume of the tetrahedron as $\frac{2}{3}c\frac{|\sin\delta_{(i)}|}{\|f_i-f\|}$, so $\frac{|\sin\delta_{(i)}|}{\|f_i-f\|}$ must be constant for all four edges of the skew parallelogram, and therefore constant over the entire net. □

Remark 3. *In the smooth setting the torsions of the asymptotic lines on a surface of constant negative Gauß curvature are constant (and opposite). Their product is equal to the Gauß curvature. Some authors define* $\frac{\sin\delta_{(i)}}{\|f_i-f\|}$ *as the torsion of the coordinate polygons [**?**] and set their product as a definition for the discrete Gauß curvature of a geometric K-net [**?**].*

One can globally scale each geometric K-net to have folding parameter equal to one, which allows it to be reconstructed from its Gauß map.

Lemma 3. *Each geometric K-net* $f : \mathbb{Z}^2 \to \mathbb{R}^3$ *(with folding parameter one) can be reconstructed from its Gauß map* $n : \mathbb{Z}^2 \to \mathbb{S}^2$ *by the relations* $f_1 - f = n_1 \times n$ *and* $f_2 - f = n \times n_2$*.*

Remark 4. *In this scaling, the Gauß map satisfies* $n \cdot n_i = \cos\delta_{(i)}$ *and the geometric K-net edge lengths are* $\|f_i - f\| = |\sin\delta_{(i)}|$*; rhombic K-nets therefore have constant and (oppositely) equal dihedral angle functions* $\delta = \delta_{(1)} = -\delta_{(2)}$*. Moreover, geometric K-nets have a maximum allowed edge length, since* $\delta_{(i)} \in \mathbb{R}$ *are considered as real valued functions.*

Remark 5. *Geometric K-nets are determined by Cauchy data on their Gauß maps, each three points on the unit sphere determine a unique fourth point by completing the spherical parallelogram [**?**], corresponding to a discrete Moutard equation restricted to the unit sphere [**?**]:*

$$n_{12} = \frac{(n_1 + n_2) \cdot n}{1 + n_1 \cdot n_2}(n_1 + n_2) - n. \tag{1}$$

We now review a second aspect of K-nets: their *algebraic* description. Using the theory of discrete integrable systems and discrete moving frames, Bobenko and Pinkall [**?**] gave a construction for *algebraic K-nets* and proved that these algebraic and geometric K-nets are in fact the same. We briefly recapitulate this construction.

We express the quaternions $\mathbb{H}$ in terms of 2x2 complex matrices as a real vector space over the Pauli matrices $\{\sigma_0, -i\sigma_1, -i\sigma_2, -i\sigma_3\}$, where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)$$

Definition 5. *We define the span of σ_0 as the space of* real quaternions *and the span of $\{-i\sigma_1, -i\sigma_2, -i\sigma_3\}$ as the space of* imaginary quaternions. *Correspondingly, each quaternion is decomposable into its* real *and* imaginary *parts.*

Throughout, we identify $\mathbb{R}^3$ with the space of imaginary quaternions.

Definition 6. *An* extended moving frame *of a family of nets is a family of maps $\Phi(\lambda) : \mathbb{Z}^2 \to \mathbb{H}^*$ from the vertices of the square lattice to the nonzero quaternions depending on a so-called spectral parameter λ, which generate a quad net f together with a Gauß map n for each $t \in \mathbb{R}$ via the following discrete analogue of the* Sym-Tafel *[**?**] or* Sym-Bobenko *[**?**] formula:*

$$f = 2\left[\Phi^{-1}\frac{\partial}{\partial t}\Phi\right]^{\mathrm{tr}=0}, \quad n = -i\Phi^{-1}\sigma_3\Phi, \quad \lambda = e^t, \quad (3)$$

where $[q]^{\mathrm{tr}=0}$ denotes the projection $\mathbb{H} \to \mathbb{R}^3$ induced by taking the quaternionic imaginary part of $q \in \mathbb{H}$, since it corresponds to the trace free part in the 2x2 matrix representation.

Definition 7. *Each extended moving frame is determined by a* Lax representation, *a pair of* Lax *(or* transition*)* matrix *maps, $X(\lambda), Y(\lambda) : \mathbb{Z}^2 \to \mathbb{H}^*$ from the edges of the square lattice to the nonzero quaternions that describe how the frame $\Phi(\lambda)$ changes, i.e., $\Phi_1 = X(\lambda)\Phi, \Phi_2 = Y(\lambda)\Phi$. The* compatibility condition *is that $\Phi_{12}(\lambda) = \Phi_{21}(\lambda)$ for all λ, which in matrices is expressed as*

$$Y_1(\lambda)X(\lambda) = X_2(\lambda)Y(\lambda). \quad (4)$$

We present a Lax representation for K-nets following [**?**]. Let $\delta_{(1)} : \mathbb{Z}^2 \to [0, \pi]$ (and $\delta_{(2)} : \mathbb{Z}^2 \to [-\pi, 0]$) be a function that depends only on the first (and second) lattice direction – as the notation suggests these will be dihedral angles of the resulting nets. Consider the Lax pair $U, V : \mathbb{Z}^2 \to \mathbb{H}$

$$U = \begin{pmatrix} \cot(\frac{\delta_{(1)}}{2})\frac{H_1}{H} & i\lambda \\ i\lambda & \cot(\frac{\delta_{(1)}}{2})\frac{H}{H_1} \end{pmatrix} \text{ and } V = \begin{pmatrix} 1 & \frac{i}{\lambda}\tan(\frac{\delta_{(2)}}{2})H_2H \\ \frac{i}{\lambda}\tan(\frac{\delta_{(2)}}{2})\frac{1}{H_2H} & 1 \end{pmatrix} \quad (5)$$

where $H = e^{ih} \in \mathbb{S}^1$ is a unit complex valued function on vertices. The resulting compatibility condition (4) is that the h variables solve the Hirota equation [**?**]

$$e^{i(h_{12}+h)} - e^{i(h_1+h_2)} = \tan\frac{\delta_{(1)}}{2}\tan\frac{\delta_{(2)}}{2}\left(1 - e^{i(h+h_1+h_{12}+h_2)}\right). \quad (6)$$

In the smooth setting the compatibility condition of the extended frame of a constant negative Gauß curvature surface is the sine-Gordon equation. Hirota's equation is known to be a discrete analogue of this equation.

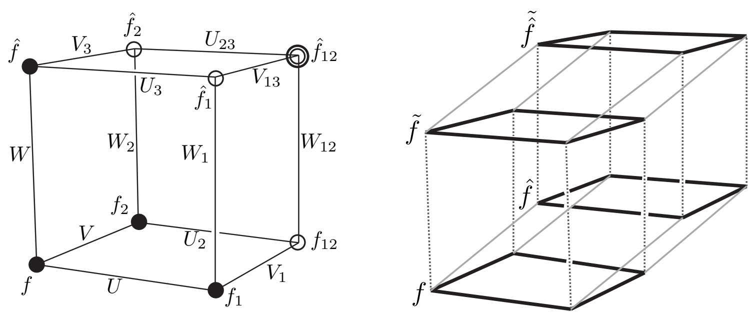

Figure 2: Left: Combinatorial 3D cube showing that the Bäcklund transformation of a K-net f to $\hat{f}$ is given by *3D consistency* of the Lax compatibility condition – the Hirota equation (6). Starting from four initial points (filled circles) a Hirota equation evolves three faces to achieve three new points (empty circles). A second evolution of Hirota on each of the resulting three faces coincide in a single point (double circle), completing a 3D cube. Right: Combinatorial 4D cube showing that 4D consistency corresponds to *Bianchi permutability* (Theorem 3) of K-net Bäcklund transformations; there exists a unique double Bäcklund transformation $\tilde{\hat{f}} = \hat{\tilde{f}}$ that is a $\hat{\alpha}$-transformation of the single $\tilde{\alpha}$-transformation $\tilde{f}$ of f and a $\tilde{\alpha}$-transformation of the single $\hat{\alpha}$-transformation $\hat{f}$ of f.

Definition 8. *An* algebraic K-net *is a member of the family of nets arising from the Lax representation* (5) *by means of differentiating its extended moving frame via the Sym formula* (3). *For a given algebraic K-net, varying λ generates the nets of its* associated family.

Theorem 1. *[**?**] Every algebraic K-net is a geometric K-net (with folding parameter equal to one), and every geometric K-net (with folding parameter equal to one) arises algebraically.*

From now on we simply refer to a *K-net*.

We now review a third aspect of K-nets: their *Bäcklund transformations* and *multidimensional consistency*. For a comprehensive review of the subject see the book by Bobenko and Suris [**?**].

A single-step Bäcklund transformation of a smooth surface of constant negative Gauß curvature is characterized by the conditions that corresponding points (i) lie in both tangent planes, (ii) are in constant distance, and (iii) that corresponding normals form a constant angle. Furthermore, asymptotic lines and curvature lines are preserved.

The discrete Bäcklund transformation for discrete K-nets in asymptotic parametrization is also characterized by these conditions and preserves the discrete asymptotic parametrization.

Theorem 2. *[**?**] Given a K-net $f : \mathbb{Z}^2 \to \mathbb{R}^3$ with Gauß map n, an angle $\alpha \neq k\pi \in \mathbb{R}$, and a direction $w \perp n_{0,0}$ there exists a unique K-net $\hat{f}$ (with*

Gauß map $\hat{n}$) such that $(\hat{f}_{0,0} - f_{0,0})$ is parallel to w, $\|\hat{f} - f\| = |\sin\alpha|$, and $\angle(n, \hat{n}) = \alpha$. The resulting K-net $\hat{f}$ is called a α-Bäcklund transformation *of f.*

K-net Bäcklund transformations can also (and perhaps more fundamentally [**?**, **?**]) be understood algebraically in terms of the *3D consistency* of the K-net Lax pair compatibility condition – the Hirota equation (6). Each quad of a K-net f together with its Bäcklund transformation $\hat{f}$ form a combinatorial cube, whose top and bottom faces correspond to f and $\hat{f}$, respectively, see Figure 2 left. It turns out that the side faces are also governed by a Hirota-type equation. If one prescribes Lax matrices W of U (or V) type (5) on the vertical edges, then the compatibility condition (4) around each face is satisfied and also of Hirota-type.

Thus, the Bäcklund transformation of a K-net can be represented by multiplying its extended frame with a matrix W of either U (or V) type. Iterating Bäcklund transformations naturally gives rise to maps $\mathbb{Z}^3 \to \mathbb{R}^3$. It turns out that 3D consistency implies *multidimensional consistency* [**?**], which for four dimensions is known as *Bianchi permutability* and given by the following theorem (see Figure 2 right).

Theorem 3. *[**?**] Consider a K-net $f : \mathbb{Z}^2 \to \mathbb{R}^3$ with Gauß map n together two Bäcklund transforms $\hat{f}$ and $\tilde{f}$ with parameters $\hat{\alpha}$ and $\tilde{\alpha}$, respectively. Then there is a unique K-net $\hat{\tilde{f}}$ that is a $\tilde{\alpha}$-Bäcklund transform of $\hat{f}$ as well as a $\hat{\alpha}$-Bäcklund transform of $\tilde{f}$.*

Remark 6. *The 3D, and resulting multidimensional, consistency of the K-net system and its corresponding Hirota-type equation can also be understood using the equation that governs its Gauß map* (1). *This is a Moutard equation restricted to the unit sphere and has been shown to be 3D and multidimensionally consistent [**?**].*

1.3 Edge-constraint nets and curvatures

Let us briefly recall the notion of edge-constraint nets and their curvatures. The definition of edge-constraint nets is first and foremost a weak coupling of a quad net with its Gauß map. This allows for a curvature theory based on normal offsets that turns out to be consistent with many known discretizations of integrable surfaces. In the case of C-nets it coincides with the definitions given in [**?**, **?**, **?**] which include the nets of constant mean curvature [**?**] and minimal nets [**?**] defined by Bobenko and Pinkall. Even nets of constant negative Gauß curvature in asymptotic line (K-nets) and curvature line parametrization (cK-nets, the topic of this paper) are in this class. Moreover, the class of edge-constraint nets also includes the associated families of all of these constant curvature nets (and furnishes them with the expected curvatures). These concepts are discussed in depth in [**?**].

Definition 9. *Let G be a quad graph. Two maps $f : G \to \mathbb{R}^3$ and $n : G \to \mathbb{S}^2$*

are said to form an edge-constraint net *if*

$$f_i - f \perp n_i + n \tag{7}$$

holds for all edges of the graph G. The map n is then called a Gauß map *for f. A unit vector $N \perp \operatorname{span}\{n_{12} - n, n_2 - n_1\}$ is said to be a face normal. A quad with face normal N has* Gauß curvature *K defined as*

$$K := \frac{\det(n_{12} - n, n_2 - n_1, N)}{\det(f_{12} - f, f_2 - f_1, N)} \tag{8}$$

and mean curvature *defined as*

$$\frac{1}{2} \frac{\det(f_{12} - f, n_2 - n_1, N) + \det(n_{12} - n, f_2 - f_1, N)}{\det(f_{12} - f, f_2 - f_1, N)}. \tag{9}$$

Remark 7. *Generically the face normal is unique up to sign but even if it is not the above defined curvatures are invariant under the choice of N (see [**?**] for more details).*

Remark 8. *This notion of curvature is motivated by the smooth Steiner formula that relates the area of offset surfaces with the curvatures of the original one. If $f^t := f + tn$ taken pointwise defines the offset surface at distance t, one finds $A(f^t) = (1 + 2\mathcal{H}t + \mathcal{K}t^2)A(f)$ where A denotes the area of the surface over a given region and $\mathcal{H}$ and $\mathcal{K}$ are the integrals of the mean and Gauß curvature of f over that region.*

Discrete curvature and asymptotic nets extend to edge-constraint nets. However, there is not always a canonical choice of Gauß map.

For C-nets, Schief [**?**, **?**] suggested unit length Gauß maps $n : \mathbb{Z}^2 \to \mathbb{S}^2$ that are *edge-parallel*, i.e., $n_i - n \parallel f_i - f$ for $i = 1, 2$. It turns out that C-nets are characterized as those quad nets with edge-parallel unit length Gauß maps [**?**]. Edge-parallel Gauß maps clearly satisfy the edge-constraint $n_i + n \perp f_i - f$.

Lemma 4. *Each C-net exhibits a 2-parameter family of edge-parallel Gauß maps, each of which extends the C-net to an edge-constraint net.*

Proof. An edge-parallel Gauß map is determined by choosing a unit vector at one vertex and then reflecting it through the perpendicular bisector plane of each edge in the C-net, i.e., for $i = 1, 2$

$$n_i = n - 2\frac{n \cdot (f_i - f)}{\|f_i - f\|^2}(f_i - f). \tag{10}$$

The composition of these reflections around each quad is the identity since its vertices are concircular. Hence, each unit length vector determines a Gauß map over the entire C-net. □

Remark 9. *As expected, the curvatures of a C-net depend on the choice of edge-parallel Gauß map.*

From now on, we assume that each C-net comes with a choice of edge-parallel Gauß map.

By definition A-nets have well defined tangent planes at each vertex given by the span of its vertex's incident edges. These planes can be used to extend an A-net to an edge-constraint net.

Lemma 5. *Each A-net extends to an edge-constraint net by choosing a unit normal at each vertex that is perpendicular to its corresponding planar vertex star, thus defining a Gauß map* $n : \mathbb{Z}^2 \to \mathbb{S}^2$.

Proof. The edge-constraint is automatically satisfied from the planar vertex star defining property of A-nets. □

From now on, we assume that each A-net comes with a choice of Gauß map.

One of the benefits of finding an extended frame (Definition 7) for a net (or family of nets) is that each immersion comes with a canonical Gauß map. For example, the Lax representation (5) allows the Gauß curvature of a K-net to be computed.

Lemma 6. *Consider a K-net (with spectral parameter* $\lambda = e^t$*). Then the Gauß curvature* (8) *is*

$$K = -2\cosh^2 t \frac{(1 - \cos\delta_{(1)} \tanh t)(1 + \cos\delta_{(2)} \tanh t)}{\cos\delta_{(1)} + \cos\delta_{(2)}} \tag{11}$$

Proof. This is a direct computation (for more discussion see [**?**]). □

Remark 10. *When the dihedral angle functions* $\delta_{(1)}$ *and* $\delta_{(2)}$ *are constant, the Gauß curvature* (11) *of a K-net (and each member of its associated family) is constant. The Gauß curvature is not constant for the most general family of K-nets that allow* $\delta_{(1)}$ *and* $\delta_{(2)}$ *to vary as functions of the first and second lattice directions, respectively. However, in the continuum limit* $\delta_{(1)}, \delta_{(2)} \to 0$ (11) *we recover* $K = -1$ *for each K-net.*

Remark 11. *Wunderlich [**?**] gave a notion of Gauß curvature for rhombic K-nets, those with equal edge length everywhere (*$\delta = \delta_{(1)} = -\delta_{(2)}$ *and* $t = 0$*), that agrees with* (11)*.*

2 Circular K-nets

2.1 A geometric definition and examples

Circular nets of constant negative Gauß curvature have been previously discussed in [**?**, **?**, **?**, **?**, **?**], where the following geometric definition is provided.

Definition 10. *A* cK-net *is a C-net (with a choice of edge-parallel Gauß map) in which each quad has the same negative Gauß curvature* $K = -\frac{1}{\rho^2}$ *for some constant* $\rho \neq 0$.

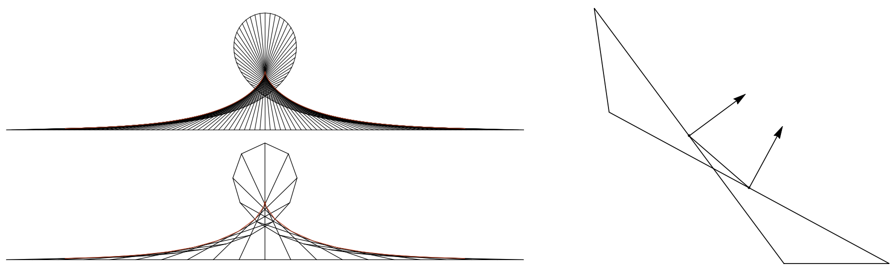

Figure 3: Left: Smooth (above) and discrete (below) Darboux transform and tractrix (red) of the straight line. Right: Natural normals for the discrete tractrix.

However, the relationship between K-nets and cK-nets, which is easily understood in the smooth setting as a reparametrization from asymptotic to curvature coordinates, remained unclear.

We begin with a few example of cK-nets. Then, we show how the Lax representation for K-nets (5) in fact extends to a Lax representation for cK-nets and that each cK-net arises in this way; moreover, the resulting Lax representation naturally gives rise to an associated family and Bäcklund transformation for cK-nets.

Example 1 (Pseudosphere)**.** *There is a natural discrete version of the tractrix construction as the curve halfway between a regular planar curve γ and its* Darboux transform $\hat{\gamma}$*, as shown in Figure 3 left. Given a regular discrete curve p (i.e., a polygon with no vanishing edges) and a starting point $\hat{p}_0$ at distance $2d > 0$ from p_0, there is a unique polygon $\hat{p}$ such that: (i) $\|\hat{p} - p\| = 2d$, (ii) $\|\hat{p}_1 - \hat{p}\| = \|p_1 - p\|$, and (iii) the quadrilaterals $p, p_1, \hat{p}_1, \hat{p}$ form planar non-embedded parallelograms (parallelograms folded along their diagonals). The regular discrete curve $\hat{p}$ is known as the* discrete Darboux transform *of p [**?**]. The* tractrix $\tilde{p}$ of p *is defined as the polygon pointwise halfway between the two: $\tilde{p} = \frac{1}{2}(\hat{p} + p)$. Normals to the smooth tractrix $\tilde{\gamma}$ are given (maybe up to sign) by normalizing the tangent vector $\hat{\gamma} - \gamma$ and rotating by 90 degrees. Similarly, we furnish the discrete tractrix $\tilde{p}$ with normals $\tilde{n}$ at vertices by taking 90 degree rotations of $\frac{\hat{p}-p}{\|\hat{p}-p\|}$, as shown in Figure 3 right.*

Starting from the polygon $p_k = (\epsilon k, 0)$ and an initial point at a given distance, say $\hat{p}_0 = (0, 2d)$ (this is the symmetric choice but that is not necessary), we generate a tractrix polygon $\tilde{p}$ (with components $(\tilde{p}_k)_x, (\tilde{p}_k)_y$ at each vertex $\tilde{p}_k$) together with normals $\tilde{n}$ that can be used to form a discrete surface of revolution: Given a rotation angle ϕ (choosing an integer fraction of 2π guarantees it closes in the rotational direction), define

$$f(k,\ell) = \begin{bmatrix} (\tilde{p}_k)_x \\ \cos(\ell\phi)(\tilde{p}_k)_y \\ \sin(\ell\phi)(\tilde{p}_k)_y \end{bmatrix} \tag{12}$$

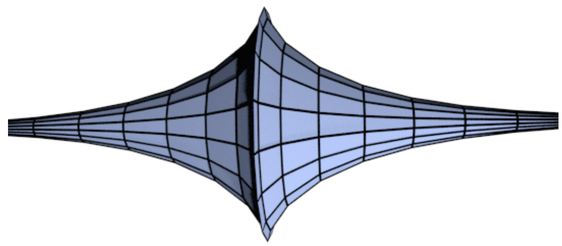

Figure 4: A cK-net pseudosphere of revolution.

and rotate the normals $\tilde{n}$ along with their corresponding points to form the Gauß map $n(k, \ell)$. By construction the quads of both f and n are planar isosceles trapezoids lying in parallel planes, so (f, n) is a circular edge-constraint net. Through elementary geometry one can compute the signed area of each isosceles trapezoid of f and its corresponding trapezoid of n. Since the face normal per quad is in fact perpendicular to each of these trapezoids, the ratio of these areas is the Gauß curvature (8), which is found to be $-\frac{1}{d^2}$ for every quad. In particular, it is independent of both discretization parameters ϵ and ϕ, so all resulting discrete pseudospheres are cK-nets.

Figure 4 shows a resulting discrete Pseudosphere.

Remark 12. *Schief [**?**] constructed by other methods the same pseudospheres as those above and provided an explicit formula for the discrete immersion. In [**?**], Bobenko, Pottmann, and Wallner, gave an implicit relation for the meridian polygon and its normals to produce discrete cK-nets of revolution. The above tractrix construction provides explicit normals at vertices furnishing an edge-constraint net of constant negative Gauß curvature:*

$$f(k,\ell) = \begin{bmatrix} \epsilon k - \tanh(\tau k) \\ \cos(\ell\phi)\,\mathrm{sech}(\tau k) \\ \sin(\ell\phi)\,\mathrm{sech}(\tau k) \end{bmatrix} \quad \text{with} \quad n(k,\ell) = \begin{bmatrix} \mathrm{sech}(\tau k) \\ \cos(\ell\phi)\tanh(\tau k) \\ \sin(\ell\phi)\tanh(\tau k) \end{bmatrix}, \tag{13}$$

where $\tau = \log \frac{2+\epsilon}{2-\epsilon}$.

Remark 13. *The radii of the circles of the quads of these pseudospheres generically varies along the axis of revolution. The squared radius of the circumcircle of quad $f(k, \ell), f(k+1, \ell), f(k+1, \ell+1), f(k, \ell+1)$ is*

$$\frac{\epsilon^2}{4} + 2\frac{\frac{\epsilon^2}{4} - \sin^2\frac{\phi}{2}}{(\frac{\epsilon^2}{4} - 1)\cosh(\tau(1+2k)) - (\frac{\epsilon^2}{4} + \cos\phi)}. \tag{14}$$

Example 2 (Trivial cK-nets: subsets of rhombic K-nets)**.** *A class of cK-nets are those given by half the vertices of a rhombic K-net. We refer to such cK-nets as* trivial *since they are completely covered by the theory of K-nets. They have been discussed from the perspective of multidimensional consistent sublattices in, e.g., [**?**, **?**].*

Consider a black-white coloring of the square grid $\mathbb{Z}^2$ and a rhombic K-net $f : \mathbb{Z}^2 \to \mathbb{R}^3$ (scaled to have folding parameter equal to one) with Gauß map $n : \mathbb{Z}^2 \to \mathbb{S}^2$. Then, the white (or black) vertices considered as a diagonal lattice of the original one $f_\circ : \mathbb{Z}^2 \to \mathbb{R}^3$ with Gauß map $n_\circ : \mathbb{Z}^2 \to \mathbb{R}^3$ is a cK-net. Indeed, each face of $f_\circ$ is formed by the neighboring vertices of a black vertex star of the rhombic K-net. Since this vertex star is planar and all edges have the same length, the resulting face of $f_\circ$ is planar with concircular vertices. The symmetry of each skew rhombus of the K-net guarantees its diagonal and corresponding Gauß map diagonal are parallel, and therefore $n_\circ$ is an edge-parallel Gauß map for $f_\circ$. Computing the Gauß curvature (8) of $f_\circ$ with edge-parallel Gauß map $n_\circ$ yields $K = -1$.

Remark 14. *It is easy to see that the pseudospheres of Example 1 are, in general, not trivial cK-nets by considering the radii of the circumcircles of each quad. Each quad of a trivial cK-net lies on a circle of radius equal to the common rhombic edge length, so all radii are equal. However, as stated in Remark 13, the radii of these circles will in general vary[2] along the pseudosphere, so they are not trivial.*

2.2 A Lax representation

In the smooth setting the curvature lines of a surface of constant negative Gauß curvature are locally the sum and difference of the arc length parametrized asymptotic lines. Therefore, it is no surprise that trivial cK-nets (see Example 2) arise by considering the diagonal lattice of a rhombic K-net – this corresponds to a discrete *global* enforcement of the relationship between asymptotic and curvature lines. However generic cK-nets, e.g., the pseudospheres of Example 1, are not trivial (see Remark 14). It is then natural to ask how generic cK-nets are related to asymptotic K-nets.

We will show that generic cK-nets also satisfy the relationship between curvature and asymptotic lines, but *locally* (along each *edge*) as opposed to *globally*. This provides a direct relationship between cK-nets and K-nets. As seen in Figure 5 (proven later in Section 2.3), each edge of a cK-net is the diagonal of a rhombus (and opposite edges are diagonals of rhombi with the same edge length), but for a generic cK-net quad its four rhombi do not share a central vertex.

We now construct an extended moving frame and Lax representation for a family of nets based on the assumption that the $\lambda = 1$ member has a (i) *rhombic factorization*, that each edge factors into the diagonal of a rhombic K-net quad and (ii) *opposite rhombic symmetry*, that opposite edges are diagonals of rhombi with the same edge length. In the subsequent Section 2.3 we will see that the $\lambda = 1$ member of the resulting family of nets is indeed a cK-net and that all

[2]When $\epsilon = -2\sin\delta$ and $\phi = 2\delta$ for some $\delta \in \mathbb{R}$ the squared radii (14) are constant and equal to $\sin^2\delta$ and the cK-net is trivial, formed from a rhombic K-net Pseudosphere given in [?].

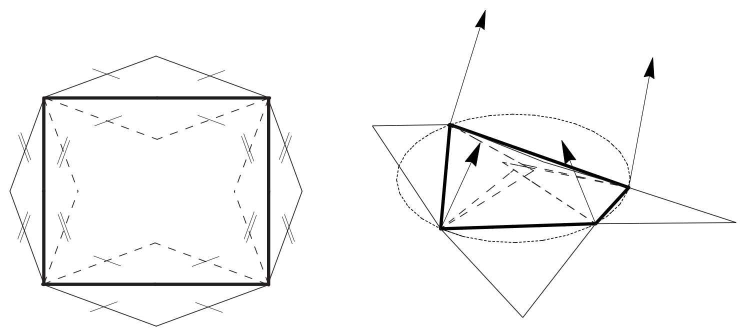

Figure 5: Left: A didactic view of a cK-net quad (thick black). Each edge is the diagonal of a rhombus and opposite edges are diagonals of rhombi with the same edge length. Generically, the four rhombi do not share a central vertex. A cK-net quad can be factored as an eight-loop of K-net edges, e.g., as the outer loop (thin black) or inner loop (dashed thin black). Right: A cK-net quad together with its normals and circumcircle. Observe that each edge is indeed the diagonal of a rhombus and the entire cK-net quad can be factored into an eight-loop of K-net edges (shown in accordance with the didactic view).

cK-nets arise this way (up to global scaling and translation), thus proving the claims just discussed.

Recalling the Lax pair (5) for K-nets and multiplying a K-net transition matrix (with equal and opposite dihedral angles) from each lattice direction yields a Lax matrix for the diagonal of a K-net skew parallelogram that is a rhombus with side length $|\sin\delta|$ when $\lambda = 1$

$$V_1 U = \begin{pmatrix} \cot\frac{\delta}{2}\frac{H_1}{H} + \tan\frac{\delta}{2}H_1 H_{12} & i(\lambda - \frac{H H_{12}}{\lambda}) \\ i(\lambda - \frac{1}{\lambda H H_{12}}) & \cot\frac{\delta}{2}\frac{H}{H_1} + \tan\frac{\delta}{2}\frac{1}{H_1 H_{12}} \end{pmatrix}, \tag{15}$$

where $H, H_{12} \in \mathbb{S}^1$ are variables now associated to the vertices of the diagonal and $H_1 \in \mathbb{S}^1$ is a variable now associated to the diagonal edge itself. Considering these *rhombic factorization* Lax matrices as the edges of a lattice and imposing the *opposite rhombic symmetry* assumption yields the following Lax representation whose compatibility condition can be satisfied.

Theorem 4. *Let $\delta_{(1)}, \delta_{(2)} : \mathbb{Z}^2 \to [-\pi, \pi]$ be functions depending only on the first and second lattice directions, respectively. Let $s : \mathbb{Z}^2 \to \mathbb{S}^1$ be a unitary complex function on vertices and let $l, m : \mathbb{Z}^2 \to \mathbb{S}^1$ be unitary complex functions on edges. Consider the Lax transition matrices*

$$\begin{aligned} L &= \begin{pmatrix} \cot\frac{\delta_{(1)}}{2}\frac{l}{s} + \tan\frac{\delta_{(1)}}{2} l s_1 & i(\lambda - \frac{s s_1}{\lambda}) \\ i(\lambda - \frac{1}{\lambda s s_1}) & \cot\frac{\delta_{(1)}}{2}\frac{s}{l} + \tan\frac{\delta_{(1)}}{2}\frac{1}{l s_1} \end{pmatrix}, \\ M &= \begin{pmatrix} \cot\frac{\delta_{(2)}}{2}\frac{m}{s} + \tan\frac{\delta_{(2)}}{2} m s_2 & i(\lambda - \frac{s s_2}{\lambda}) \\ i(\lambda - \frac{1}{\lambda s s_2}) & \cot\frac{\delta_{(2)}}{2}\frac{s}{m} + \tan\frac{\delta_{(2)}}{2}\frac{1}{m s_2} \end{pmatrix}. \end{aligned} \tag{16}$$

Then the compatibility condition (4) *can be satisfied, yielding an evolution equation for the variables. Each net in the* associated family *depending on the spectral parameter* $\lambda = e^t, t \in \mathbb{R}$, *arising from the corresponding extended moving frame of Definition 7 integrated via the Sym-Bobenko formula* (3)*, is edge-constraint.*

Proof. Given s, s_1, s_2, l, m, and setting $\mathsf{t}_{(i)} = \tan\frac{\delta_{(i)}}{2}$ for notational simplicity, one finds after a long computation that the compatibility condition is satisfied when

$$\begin{aligned}
l_2 &= \frac{-m\mathsf{t}_{(2)}(ss_2 + \mathsf{t}_{(1)}^2) + l\mathsf{t}_{(1)}(ss_2 + \mathsf{t}_{(2)}^2)}{m(-l\mathsf{t}_{(2)}(1 + ss_2\mathsf{t}_{(1)}^2) + m(\mathsf{t}_{(1)} + ss_2\mathsf{t}_{(1)}\mathsf{t}_{(2)}^2))}\\
m_1 &= \frac{m\mathsf{t}_{(2)}(ss_1 + \mathsf{t}_{(1)}^2) - l\mathsf{t}_{(1)}(ss_1 + \mathsf{t}_{(2)}^2)}{l(l\mathsf{t}_{(2)}(1 + ss_1\mathsf{t}_{(1)}^2) - m(\mathsf{t}_{(1)} + ss_1\mathsf{t}_{(1)}\mathsf{t}_{(2)}^2))}\\
s_{12} &= s\frac{S^+ - S_1^-}{S^+ - S_2^-}, \quad \text{where}\\
&\quad S^+ = l^2\mathsf{t}_{(1)}\mathsf{t}_{(2)}(1 + ss_1\mathsf{t}_{(1)}^2)(ss_2 + \mathsf{t}_{(2)}^2) + m^2\mathsf{t}_{(1)}\mathsf{t}_{(2)}(ss_1 + \mathsf{t}_{(1)}^2)(1 + ss_2\mathsf{t}_{(2)}^2),\\
&\quad S_1^- = lm(ss_1\mathsf{t}_{(1)}^2 + \mathsf{t}_{(2)}^2(2\mathsf{t}_{(1)}^2 + ss_2(1 + 2ss_1\mathsf{t}_{(1)}^2 + \mathsf{t}_{(1)}^4)) + ss_1\mathsf{t}_{(1)}^2\mathsf{t}_{(2)}^4),\\
&\quad S_2^- = lm(ss_2\mathsf{t}_{(1)}^2 + \mathsf{t}_{(2)}^2(2\mathsf{t}_{(1)}^2 + ss_1(1 + 2ss_2\mathsf{t}_{(1)}^2 + \mathsf{t}_{(1)}^4)) + ss_2\mathsf{t}_{(1)}^2\mathsf{t}_{(2)}^4).
\end{aligned} \tag{17}$$

That each net in the associated family are edge-constraint follows by construction. Each edge is the diagonal of a folded parallelogram (folded rhombus when $\lambda = 1$) with vertex planes perpendicular to n at incident vertices. Since folded parallelograms have 180° rotational symmetry, the edge-constraint is satisfied. □

Remark 15. *Unlike for the case of K-nets where the compatibility condition is a known discrete sine-Gordon equation (the Hirota equation* (6)*), the authors were not able to explicitly relate the evolution* (17) *to the sine-Gordon equation. However, we can gauge the Lax matrices* L *and* M *to only have edge variables: with*

$$G = \begin{pmatrix} \sqrt{s} & 0 \\ 0 & \sqrt{s}^{-1} \end{pmatrix}$$

we find

$$G_1^{-1}LG = \begin{pmatrix} l\frac{\cot\frac{\delta_{(1)}}{2}}{\sqrt{s_1 s}} + l\tan\frac{\delta_{(1)}}{2}\sqrt{s_1 s} & i\frac{\lambda}{\sqrt{s_1 s}} - i\frac{\sqrt{s_1 s}}{\lambda} \\ i\lambda\sqrt{s_1 s} - i\frac{1}{\lambda\sqrt{s_1 s}} & \frac{1}{l}\cot\frac{\delta_{(1)}}{2}\sqrt{s_1 s} + \frac{1}{l}\frac{\tan\frac{\delta_{(1)}}{2}}{\sqrt{s_1 s}} \end{pmatrix}$$

and a similar expression for $G_2^{-1}MG$*. For* $i = 1, 2$ *one can set* $\sqrt{s_i s}$ *as new variables on the edges. These variables are then the discrete sine-Gordon variables introduced by Bobenko and Pinkall [**?**], but taken from a 4D lattice and* (17) *yields the corresponding evolution equation (multiply the evolution of* s_{12}

by s_1 or s_2, respectively). The relationship to 4D K-net compatibility is discussed further in Remark 18.

Note that the functions $\delta_{(1)}, \delta_{(2)}$ are now functions of only the L and M lattice directions, respectively. As suggested by the notation, however, they still correspond to the dihedral angles of the factorizing parallelograms of an edge of one of the resulting nets, which are by assumption equal for parallelograms corresponding to opposite edges. As noted in Remark 4 K-net parallelograms have a maximum allowed edge length given by the restriction that the dihedral angle functions must be real valued to guarantee the K-net lives in $\mathbb{R}^3$. It is therefore expected that the family of nets arising from Theorem 4 would also have a maximum edge length. However, since the corresponding K-net parallelograms are *virtual* in the Lax representation (16) – they are never constructed explicitly – one can algebraically extend the Lax variables so that no geometric restrictions arise.

Theorem 5. *Let $\delta_{(1)}, \delta_{(2)} : \mathbb{Z}^2 \to \mathbb{C}$ be complex valued functions depending only on the first and second lattice directions, respectively, such that $\tan \frac{\delta_{(i)}}{2}$ is either real or unitary. Let $s = e^{i\rho} : \mathbb{Z}^2 \to \mathbb{S}^1$ be a unitary complex function on vertices and let $l, m : \mathbb{Z}^2 \to \mathbb{C}$ be complex valued functions on edges with absolute value ($i = 1$ for l and $i = 2$ for m)*

$$\sqrt{\frac{\cos\left(\rho_i - \arg\tan\frac{\delta_{(i)}}{2}\right) + \cos\left(\rho - \arg\tan\frac{\delta_{(i)}}{2}\right)}{\cos\left(\rho_i + \arg\tan\frac{\delta_{(i)}}{2}\right) + \cos\left(\rho + \arg\tan\frac{\delta_{(i)}}{2}\right)}}. \tag{18}$$

Then the Lax matrices (16) of Theorem 4 satisfy the compatibility condition with the same evolution of variables Equation (17) and the resulting nets are edge-constraint.

Proof. The length of a virtual K-net edge is given (up to sign) by $e_i = \sin \delta_{(i)}$. If $e_i > 1$ then $\delta_{(i)} = \arcsin e_i$ is complex and $\tan \frac{\delta_{(i)}}{2}$ goes from real to unitary. In the case that these tangents are unitary, the structure of the Lax matrices changes and, in general, no longer corresponds to a quaternion – a basic assumption of the construction. Computing the condition that the resulting matrices are in fact quaternions, one finds it is equivalent to the absolute value of the l, m edge variables be equal to Equation (18). Note that this value is indeed one, so l, m are unitary when $\tan \frac{\delta_{(i)}}{2}$ and $\delta_{(i)}$ are real.

The computation that the compatibility condition is satisfied by the evolution (17) did not use the assumptions on the variables, so it is still valid.

The geometric argument based on the symmetry of the K-net parallelograms no longer holds to prove that the resulting nets are edge-constraint (since this parallelogram is not in $\mathbb{R}^3$), but the edge-constraint can be verified algebraically using the Sym-Bobenko formula (3). □

Remark 16. *For complex valued $\delta_{(i)}$ functions these L (or M) Lax matrices still factor into a product of U, V matrices of the K-net form (5), but each factor*

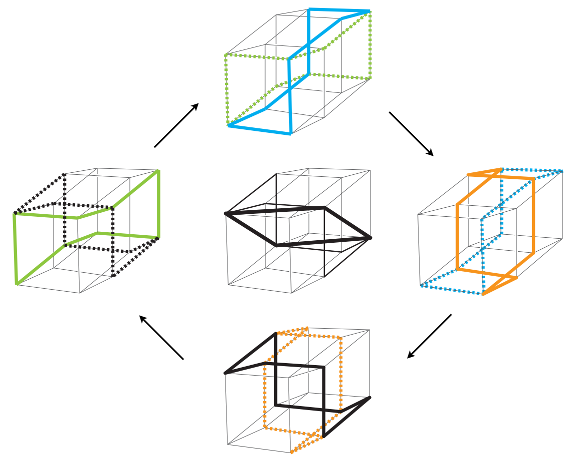

Figure 6: The relationship between K-nets and the Lax representation (16) is given by evolving the 8-loop corresponding to a generic quad. In the center, the diagonal quad (thick black) is shown with its corresponding 8-loop in the K-net 4D compatibility cube. Evolving a Hirota-type equation on each face of the initial loop closes to the K-net 4D consistency cube. The evolution goes bottom to left to top to right to bottom and is also highlighted by the color scheme black to green to blue to orange to black. In each 4D consistency cube the previous 8-loop is shown in dotted line and the current is shown in solid.

is no longer a quaternion, but a biquaternion. Recall that the biquaternions are given as the complex *vector space over the Pauli matrices* (2)*. We refer to both quaternionic and biquaternionic matrices of this form as K-net matrices. This algebraic extension of the variables in the product of two K-net Lax matrices (with opposite parameters) is actually well known. In the smooth setting it corresponds to the existence of so-called* Breather *solutions of the sine-Gordon equation, where Bäcklund transformations of opposite complex parameters are used to generate surfaces arbitrarily far away from the original one [**?**].*

Remark 17. *The variables of Theorem 5 include the restricted version of Theorem 4, so from now on we only consider this extended version of the Lax representation.*

Remark 18. *The L, M Lax matrix compatibility is closely related to the 4D consistency of the K-net system on a 4D cube as shown in Figure 2 right. The relationship is highlighted by Figure 6. The 2D compatibility condition of* (16) *corresponds to an 8-loop of K-net edges on such a 4D cube since its edges are given by diagonals on the 2D faces. In general, an arbitrary K-net 8-loop will not result in the identity for all λ. However, since the compatibility condition* (16) *can be satisfied for all λ, evolving the original K-net 8-loop by completing*

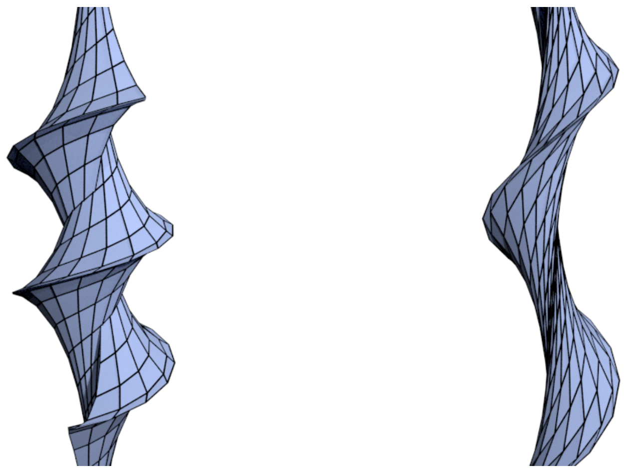

Figure 7: Two members of the associated family of the pseudosphere. A closed form expression for this family is given by (35).

each face using a Hirota-type equation results in a 4D consistent K-net cube. Note that in general the 4D solution cannot be extended from a 3D system as discussed in Figure 5.

Since we can choose the spectral parameter $\lambda = e^t$ freely, the above Lax pair of Theorem 5 gives rise to a one parameter *associated family* of nets, examples of which are shown in Figure 7.

2.3 cK-nets and their associated families from the Lax representation

We now investigate some geometric properties of the family of edge-constraint nets generated by Theorem 5. In particular, we see that the resulting $\lambda = 1$ net is a circular net with edge-parallel Gauß map of constant negative Gauß curvature $K = -1$, i.e., a cK-net. Moreover, every cK-net arises from this algebraic construction.

Theorem 6. *The net $f : \mathbb{Z}^2 \to \mathbb{R}^3$ with Gauß map $n : \mathbb{Z}^2 \to \mathbb{R}^3$ that arises from the system (16) with algebraic parameters as in Theorem 5 has Gauß curvature $K = -1$ for all values of the spectral parameter $\lambda = e^t$. It is circular with an edge-parallel Gauß map if $\lambda = 1$.*

Proof. The proof is a direct calculation. Solve the system for one quad and look at the curvature as well as the quaternionic cross-ratio: The net and its Gauß map are given by the Sym formula (3). The curvature can then be computed by (8). When $\lambda = 1$, one can compute that $f_i - f \parallel n_i - n$ directly. To show the circularity of each quad f, f_1, f_{12}, f_2 one can utilize the fact that the cross-ratio $cr = \frac{a-b}{b-c}\frac{c-d}{d-a}$ of four complex numbers a, b, c, and d is real if and only if the points are concircular. For four points in $\mathbb{R}^3$ given as imaginary quaternions A, B, C, and D this translates into $(A-B)(B-C)^{-1}(C-D)(D-A)^{-1} \in \mathbb{R}$ – see, e.g., [**?**]. This quantity can be computed to be real for $\lambda = 1$. □

Remark 19. *When $\delta_{(1)}$ and $\delta_{(2)}$ (as functions of the L and M lattice directions, respectively) are not constant, the resulting nets still have $K=-1$. This is in contrast to K-nets, where the Gauß curvature is not constant when $\delta_{(1)}$ and $\delta_{(2)}$ (as functions of the U and V lattice directions, respectively) are not constant (see Remark 10).*

Remark 20. *While in the K-net case all members of the associated family are A-nets (in discrete asymptotic line parametrization) here we have C-nets (discrete curvature line parametrization) when $\lambda = 1$. Other values of λ still give rise to edge-constraint nets of constant negative Gauß curvature (as shown in Theorem 6), but in general the quadrilaterals are no longer planar. This is expected since, in the smooth setting, in asymptotic parametrization the associated family maps $\|f_x\| \to \lambda \|f_x\|$ and $\|f_y\| \to \frac{1}{\lambda}\|f_y\|$ and only for strict Chebyshev parametrization $\|f_x\| = \|f_y\|$ do $f_x \pm f_y$ give rise to curvature directions everywhere.*

In fact, up to global scaling and translation, every cK-net arises from the Lax representation (16) with $\lambda = 1$. It is enough to show that each quad of a cK-net arises this way, and since it is only defined up to a global scaling, we restrict to $K=-1$.

Theorem 7. *Each circular net with an edge-parallel Gauß map (f, n) of Gauß curvature $K=-1$ arises from the Lax representation (16) with algebraic parameters as in Theorem 5 and $\lambda = 1$.*

Proof. The proof follows in two steps: (i) we show that each cK-net quad is uniquely determined from *Cauchy data*, i.e., given a triple of vertices $f, f_1, f_2 \in \mathbb{R}^3$ and unit vector n (that generates an edge-parallel Gauß map) a fourth concircular vertex f_{12} is uniquely determined by the condition $K=-1$, and (ii) we show that these Cauchy data determine two edges in the Lax representation (16) with variables as in Theorem 5. Evolving the compatibility condition (17) and integrating via Sym-Bobenko as in Theorem 6 (with $\lambda = 1$) then uniquely determines a $K=-1$ cK-net quad whose fourth vertex must be f_{12}, since it is determined uniquely from the same Cauchy data as in Step (i).

Step (i): Each circular quad with edge-parallel Gauß map can be written (up to translation and rotation) by its vertices $f = re^{i\phi}$, $f_1 = re^{i\phi_1}$, and $f_2 = re^{i\phi_2}$ lying on a circle of radius r centered at the origin lying in the x, y-plane, which we identify with $\mathbb{C}$, together with an initial normal $n = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$ that generates its edge-parallel Gauß map. The Gauß curvature (8) can then be found to be

$$K = \frac{\sin^2(\alpha)}{r^2} \frac{\sin\left(2(\beta - \phi) + \frac{1}{2}(\phi - \phi_1 + \phi_{12} - \phi_2)\right)}{\sin\left(\frac{1}{2}(\phi - \phi_1 + \phi_{12} - \phi_2)\right)}. \tag{19}$$

Since $K=-1$, we can solve for ϕ_{12}:

$$\phi_{12} = -2\arctan\left(\frac{\sin\left(2(\beta-\phi)+\frac{1}{2}(\phi-\phi_1-\phi_2)\right)+r^2\csc^2(\alpha)\sin\left(\frac{1}{2}(\phi-\phi_1-\phi_2)\right)}{\cos\left(2(\beta-\phi)+\frac{1}{2}(\phi-\phi_1-\phi_2)\right)+r^2\csc^2(\alpha)\cos\left(\frac{1}{2}(\phi-\phi_1-\phi_2)\right)}\right) \tag{20}$$

which gives

$$f_{12} = re^{-i(\phi-\phi_1-\phi_2)}\frac{r^2+\sin^2(\alpha)e^{-2i(\beta-\phi)}}{r^2+\sin^2(\alpha)e^{2(\beta-\phi)i}}. \tag{21}$$

So, given three initial points and a normal at the middle one there is a unique fourth vertex and unique edge-parallel Gauß map that furnish a circular net with Gauß curvature $K=-1$.

Step (ii): We show that three points and a unit vector (that extends to an edge-parallel Gauß map) determine the Lax variables (of Theorem 5) along two edges. The argument is similar for each edge, so we argue using a generic shift $i=1,2$. Determining a Lax matrix of Theorem 5 is equivalent to specifying the following variables: a dihedral angle variable $\delta_{(i)}$ that is either real, or such that $\tan\frac{\delta_{(i)}}{2}$ is unitary, two vertex variables $s_i = e^{i\rho_i}$ (that is determined from the geometry and an arbitrary $s=e^{i\rho}$), and the argument of a complex valued edge variable l or m (its absolute value must be equal to (18)).

We start by working backwards, let $f_i - f$ be an edge of a cK-net arising from Theorem 6 with $\lambda = 1$. Denote its edge length by $d>0$ and let θ be the angle it makes with each of its incident normals n, n_i (these angles are equal since $n_i - n \parallel f_i - f$). Then we find the following relationships between the Lax variables and these geometric quantities:

$$\begin{aligned} \sin^2\delta_{(i)} &= \frac{d^2}{4}+\cos^2\theta, \\ \cos(\theta) &= -\sin(\frac{\rho+\rho_i}{2})\sin\delta_{(i)}, \quad \text{and} \\ d^2 &= 4\cos^2(\frac{\rho+\rho_i}{2})\sin^2\delta_{(i)} \end{aligned} \tag{22}$$

Solving these equations for the Lax variables we find:

$$\begin{aligned} \delta_{(i)} &= \arcsin\sqrt{\frac{d^2}{4}+\cos^2\theta} \quad \text{and} \\ \rho_i &= -\rho + 2\arcsin(-\cos(\theta)\csc(\delta_{(i)})). \end{aligned} \tag{23}$$

Hence, the dihedral angle and vertex based variables of the Lax matrices are determined from the geometric data of an arbitrary edge and its edge-parallel Gauß map.

The final degree of freedom in the Lax matrix is the edge variable l (or m). Its absolute value is determined by (18), but we need to verify that this radicand is indeed nonnegative from arbitrary geometric data. We find the following equivalent inequalities:

$$
\begin{aligned}
& 0 \le \frac{\cos(\frac{\rho+\rho_i}{2} - \arg\tan\frac{\delta_{(i)}}{2})}{\cos(\frac{\rho+\rho_i}{2} + \arg\tan\frac{\delta_{(i)}}{2})} \\
\iff\ & 0 \le d^2 \cos^2(\arg\tan\frac{\delta_{(i)}}{2}) - \cos^2\theta \sin^2(\arg\tan\frac{\delta_{(i)}}{2}) \\
\iff\ & 0 \le 4\sin^2\theta.
\end{aligned} \tag{24}
$$

The last inequality (found using that $\sec^2(\arg\tan\frac{\delta_{(i)}}{2}) = \sin^2\delta_{(i)}$) is clearly satisfied, so the radicand is nonnegative. The argument of the edge variable l (or m) encodes the rotation of the corresponding edge in $\mathbb{R}^3$ about one of its incident normals. Each edge together with its edge-parallel Gauß map (anchored at their corresponding vertices) lie in a plane. Given a target plane, we can choose this argument so that the plane arising from the Lax representation aligns with this target one.

Therefore, given Cauchy data, three vertices f, f_1, f_2 and a unit vector n (that extends to an edge-parallel Gauß map), the formulas (23) determine the dihedral angle and vertex based Lax variables of the matrices (up to a global initial choice of ρ) and the argument of l (or m) can be chosen to correctly align the planes spanned by each edge together with its edge-parallel Gauß map (anchored at their corresponding vertices).

The two unique evolutions given in *Step (i)* and *Step (ii)* are from the same data, so their results coincide. □

2.4 Bäcklund transformations of cK-nets

In this section we introduce a Bäcklund transformation for discrete surfaces of constant negative Gauß curvature in curvature line parametrization (cK-nets). They are characterized by the same geometric conditions as K-nets, arise through a similar algebraic construction (see the discussion around Theorem 2), and preserve the discrete curvature line parametrization. The Bäcklund transformation also carries over to the associated family of cK-nets, generating edge-constraint nets of constant negative Gauß curvature in more general types of parametrizations.

Algebraically, the frame $\tilde{\Phi}$ of a single Bäcklund transformation $(\tilde{f}, \tilde{n})$ of a cK-net (f, n) is given by multiplying its frame Φ with transition matrices W of the algebraic structure of one of the U, V K-net matrices (5). These W depend on the given spectral parameter λ and lattice variables of Φ (namely, s, l, m) and contain as degrees of freedom the Bäcklund parameter α and a vertex lattice variable $\tilde{s}$ that will be the corresponding vertex lattice variable of the resulting frame $\tilde{\Phi}$. One can understand these W as transition matrices along a third lattice direction that connects the initial frame Φ and its Bäckland transformation $\tilde{\Phi}$ (recall the K-net Bäcklund transformation construction shown in Figure 2).

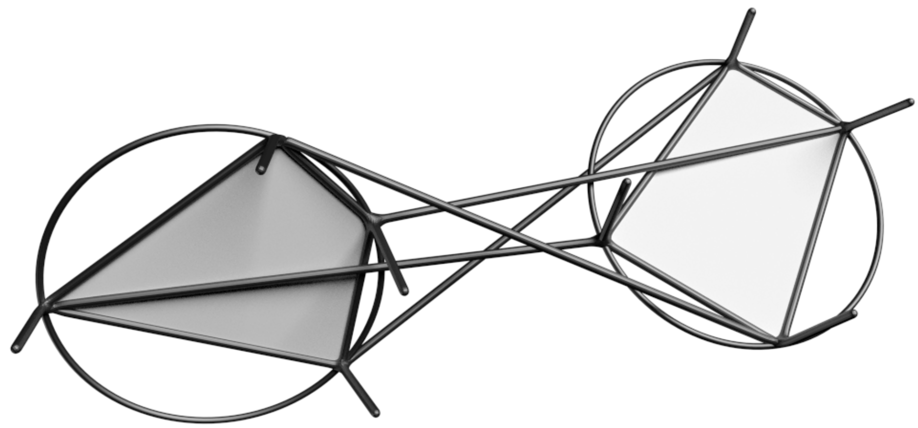

Figure 8: A cK-net quad with its Bäcklund transform (with their respective edge-parallel Gauß maps). The corresponding 3D compatibility cube has cK-net quads on the top and bottom faces (satisfying equation (17)), but skew parallelograms (satisfying equation (26)) on each side face. This is unusual for discrete Bäcklund transforms, which usually have an equation of the same type on each face [**?**].

Explicitly, we have $\tilde{\Phi}(k,\ell,\lambda) = W(\alpha,\lambda)\Phi(k,\ell,\lambda)$ with

$$W(\alpha,\lambda) = \begin{pmatrix} \cot(\frac{\alpha}{2})\frac{\tilde{s}}{s} & i\lambda \\ i\lambda & \cot(\frac{\alpha}{2})\frac{s}{\tilde{s}} \end{pmatrix}. \tag{25}$$

The resulting frame $\tilde{\Phi}$ can then be integrated via the Sym-Bobenko formula (3). As each cK-net extended frame factors into a sequence of K-net matrices, existence of Bäcklund transformations and a Bianchi permutability theorem follow from the corresponding theorems for K-nets (Theorems 2 and 3) that guarantee multidimensional consistency.

Theorem 8. *Let $f : \mathbb{Z}^2 \to \mathbb{R}^3$ with Gauß map n and Gauß curvature $K = -1$ be a cK-net. Then (up to a global degree of freedom fixing an initial normal)*

1. *For every angle $\alpha \neq 0 \in (-\pi, \pi)$ there exists a unique cK-net $\tilde{f}$ with Gauß map $\tilde{n}$ such that $\|\tilde{f} - f\| = \sin\alpha$, $\angle(n, \tilde{n}) = \alpha$, and $(\tilde{f} - f) \perp n, \tilde{n}$. The nets $\tilde{f}$ are called the* Bäcklund transforms *of f.*

2. *For every pair of Bäcklund transforms $\hat{f}$ and $\tilde{f}$ with parameters $\hat{\alpha}$ and $\tilde{\alpha}$, respectively, there exists a unique cK-net $\hat{\tilde{f}}$ that is a $\tilde{\alpha}$-Bäcklund transform of $\hat{f}$ as well as a $\hat{\alpha}$-Bäcklund transform of $\tilde{f}$.*

Proof. The result follows from the corresponding theorem for asymptotic K-nets by factoring the cK-net frame into a sequence of K-net matrices.

For completeness we now describe the evolution equations for the Lax matrix variables s, l, m to $\tilde{s}, \tilde{l}, \tilde{m}$. For Bäcklund matrices W, solving $W_1 L = \tilde{L} W$ and $W_2 M = \tilde{M} W$ yields (with $\mathsf{t}_{(i)} = \tan \frac{\delta_{(i)}}{2}$)

$$\begin{aligned}
\tilde{s}_1 &= \frac{\tilde{s}\sin(\alpha)\left(ss_1 + \mathsf{t}_{(1)}^2\right) - l\mathsf{t}_{(1)}\left((ss_1 - 1)\cos(\alpha) + ss_1 + 1\right)}{l\left(l\sin(\alpha)\left(ss_1\mathsf{t}_{(1)}^2 + 1\right) + \tilde{s}\mathsf{t}_{(1)}\left((ss_1 - 1)\cos(\alpha) - ss_1 - 1\right)\right)} \\
\tilde{l} &= \frac{s\left(\tilde{s} - l\mathsf{t}_{(1)}\cot\left(\frac{\alpha}{2}\right)\right)}{l - \tilde{s}\mathsf{t}_{(1)}\cot\left(\frac{\alpha}{2}\right)} \\
\tilde{s}_2 &= \frac{\tilde{s}\sin(\alpha)\left(ss_2 + \mathsf{t}_{(2)}^2\right) - m\mathsf{t}_{(2)}\left((ss_2 - 1)\cos(\alpha) + ss_2 + 1\right)}{m\left(m\sin(\alpha)\left(ss_2\mathsf{t}_{(2)}^2 + 1\right) + \tilde{s}\mathsf{t}_{(2)}\left((ss_2 - 1)\cos(\alpha) - ss_2 - 1\right)\right)} \\
\tilde{m} &= \frac{s\left(\tilde{s} - m\mathsf{t}_{(2)}\cot\left(\frac{\alpha}{2}\right)\right)}{m - \tilde{s}\mathsf{t}_{(2)}\cot\left(\frac{\alpha}{2}\right)}
\end{aligned} \tag{26}$$

That $\tilde{s}_{12} = \tilde{s}_{21}$ follows from $s_{12} = s_{21}$. Therefore, up to an initial choice of $\tilde{s} = e^{i\theta}$ at one point fixing an initial normal vector, the evolution is uniquely determined. The immersion and Gauß map of the transform are given by

$$\begin{aligned}
\tilde{f}(k,\ell) &= f(k,\ell) + \sin(\alpha)\left[\Phi^{-1}(k,\ell)\begin{pmatrix} 0 & i\frac{s(k,\ell)}{\tilde{s}(k,\ell)} \\ i\frac{\tilde{s}(k,\ell)}{s(k,\ell)} & 0 \end{pmatrix}\Phi(k,\ell)\right]^{\mathrm{tr}=0} \quad \text{and} \\
\tilde{n}(k,\ell) &= -i\Phi^{-1}(k,\ell)W^{-1}(k,\ell)\sigma_3 W(k,\ell)\Phi(k,\ell),
\end{aligned} \tag{27}$$

so from an initial cK-net we only require the $\tilde{s}$ variables to describe its Bäcklund transform. □

Remark 21. *Figure 8 shows a cK-net quad and a Bäcklund transform (together with their corresponding Gauß maps), forming a 3D compatibility cube. Usually, 3D compatibility cubes have equations of the same type on each face (see the discussion around Theorem 2) leading to the notion of multidimensional consistency [**?**]. However, for cK-nets, the equation on the side quadrilaterals* (26) *is different from that on the top and bottom pair of quadrilaterals* (17)*; geometrically one sees that the sides are skew parallelograms while the top and bottom are circular quads.*

Remark 22. *The Bäcklund transformation works for nets* (f^λ, n^λ) *in the associated family with spectral parameter* $\lambda = e^t$*, giving rise to more generally parametrized edge-constraint nets of constant negative Gauß curvature. The equations are given by* (27)*, with* (f, n) *replaced by* (f^λ, n^λ)*, the frame* Φ *replaced with* Φ^λ*, and the constant distance* $\sin\alpha = \|\tilde{f} - f\|$ *replaced by* $\frac{\sin\alpha}{\cosh t - \cos\alpha \sinh t} = \|\tilde{f}^\lambda - f^\lambda\|$*. Also, the constant angle between normals is given by* $\arccos\frac{\cos\alpha\cosh t - \sinh t}{\cosh t - \cos\alpha\sinh t}$*, so the relationship* $\|\tilde{f}^\lambda - f^\lambda\|^2 + (\tilde{n}\cdot n)^2 = 1$ *still holds.*

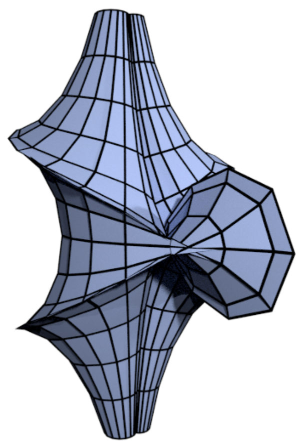

Figure 9: A discrete circular Kuen surface. The coordinate polygons in one direction are planar.

Therefore, just as in the smooth setting, the Bäcklund parameter and spectral parameter are independent of each other, but can be used to generate a discrete surface in different parametretrizations: a transformed surface at distance $\sin\beta$ *with dot product between normals given by* $\cos\beta$ *can be found in discrete curvature line parametrization (as a cK-net) by setting* $\lambda = 1$ *and* $\alpha = \beta$*, or in more general parametrizations, by first fixing a Bäcklund parameter* α *and then solving* $\|\tilde{f}^\lambda - f^\lambda\| = \frac{\sin\alpha}{\cosh t - \cos\alpha \sinh t} = \sin\beta$ *for the spectral parameter* $\lambda = e^t$*. In general, setting* $\lambda = 1$ *and varying* α *generates a family of cK-net surfaces, conversely, fixing* α *and varying* λ *generates a similar family of surfaces, but in more general parametrizations. For an explicit example see Figure 11 and the remarks after Theorem 9.*

Example 3. *The construction of the Pseudosphere given at the start of Setion 2 is in fact a Bäcklund transformation of the straight line (details can be found in Section 2.5.1). Figure 9 shows a discrete Kuen surface; it arises as a Bäcklund transformation of the Pseudosphere. Shown in Figure 1 are the Bäcklund transformations of the Pseudosphere aligned by their angular parameter. Since the Bäcklund transformation is invertible one finds both the straight line and the Kuen surface therein.*

2.4.1 Double Bäcklund transformations and a remark on multidimensional consistency

As discussed in Remark 21 and shown in Figure 8 the 3D compatibility cube formed by a cK-net quad with its Bäcklund transformation does not have an equation of the same type on each face. However, certain types of double

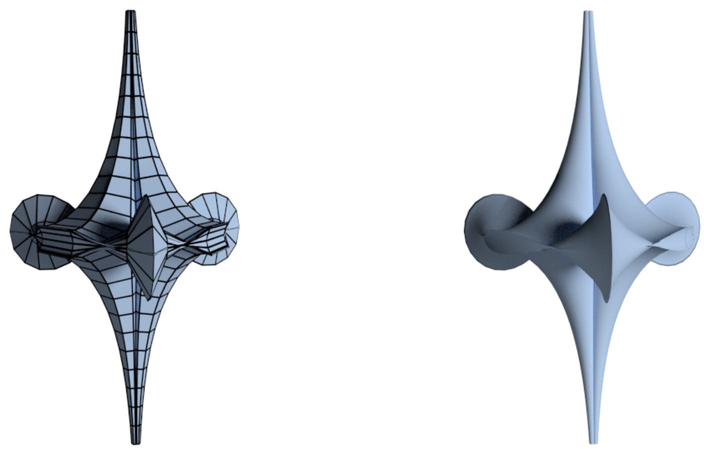

Figure 10: A nonfactorizable double Bäcklund transform of the vacuum sided (left) with the corresponding analytic solution (right). The coordinate polygons in one direction are planar.

Bäcklund transformations of cK-nets generate familiar 3D consistent cubes that agree with those studied by Schief [**?**].

Lemma 7. *Let f be a cK-net. If $\tilde{f}$ is a double Bäcklund transform with parameters α and $-\alpha$ then the resulting 3D compatibility cube formed by a quad of f and $\tilde{f}$ is a 3D consistent cube with circular quads on all sides.*

Proof. A single Bäcklund transformation is given by multiplying the extended frame of f by a K-net matrix (5), so a double transformation with opposite parameters corresponds to multiplying by two K-net matrices with opposite parameters, which is a rhombic cK-net Lax matrix (16). Therefore, after integrating via the Sym-Bobenko formula (with $\lambda = 1$), the resulting side faces will also be circular. □

This observation immediately gives the following result.

Corollary 1. *The Kuen surface that arises from the Pseudosphere with parameter $\frac{\pi}{2}$ has planar coordinate polygons in one lattice direction.*

Proof. Since the Pseudosphere arises as a Bäcklund transform from the straight line with parameter $-\frac{\pi}{2}$, the Kuen surface can be viewed as a double Bäcklund transform of the line that is formed by cubes with circular sides. Thus the line and all parameter polygons of the Kuen net in one direction are sides of a strip of circular quadrilaterals, which clearly must be planar (since it contains the straight line in its border). □

Double Bäcklund transformations with real parameters α and $-\alpha$ as above can clearly be represented by multiplying the frame by a Lax matrix of the type of L or M (see the discussion around (15)), factorizable into a product of quaternionic K-net matrices. However, recall (see Theorem 5) that the matrices L, M are more general and allow for α to be complex valued. Geometrically one can think of this as follows: For a single Bäcklund transformation the distance of the transformed points to their preimages in $\mathbb{R}^3$ must be $\sin\alpha$. Thus, if the distance is larger than one, the angle α is no longer real and so neither is the transformed surface. However with a second Bäcklund transformation one can achieve a real solution again. These double Bäcklund transformations that do not factor into two real ones are in fact well known in the continuous case [**?**]. Figure 10 shows an example, a so-called *breather* solution (the name stems from the behavior of the corresponding solution to the sine-Gordon equation).

Schief has described cK-net Bäcklund transformations in [**?**] in the realm of reductions of circular nets but was not able to get the single step transformations since they do not give rise to circular 3D compatibility cubes.

2.5 Explicit discrete parametrizations for transformations of the straight line

We now present closed form equations for some Bäcklund transformations of the straight line, together with their associated families. As in the smooth setting, single transformations give rise to Beltrami's pseudosphere (Figure 7) and Dini's surfaces (Figure 11), while double transformations (with opposite parameters $\pm\alpha$) give rise to breather surfaces (Figure 10) and Kuen's surface (Figure 9).

2.5.1 Discrete Dini's surfaces

The straight line (notated with a zero subscript) can be represented by edge and vertex functions $l_0(k,\ell) = m_0(k,\ell) = s_0(k,\ell) = (-1)^\ell$ plugged into the Lax matrices (16) and integrated via the Sym-Bobenko formula (3). For simplicity throughout this section we assume that the parameter line functions $\delta_{(1)}(k), \delta_{(2)}(\ell)$ are constants. Explicitly, the corresponding immersion and Gauß map (f_0, n_0) with spectral parameter $\lambda = e^t$ are given by

$$f_0(k,\ell,t) = \begin{bmatrix} -2x(k,\ell,t) \\ 0 \\ 0 \end{bmatrix} \quad \text{with} \quad n_0(k,\ell,t) = \begin{bmatrix} 0 \\ \Im\omega(k,\ell,t) \\ \Re\omega(k,\ell,t) \end{bmatrix}, \text{where}$$

$$x(k,\ell,t) = k\left(\frac{\cosh t \sin\delta_{(1)}}{1+\sinh^2 t \sin^2\delta_{(1)}}\right) + \ell\left(\frac{\sinh t \sin\delta_{(2)} \cos\delta_{(2)}}{1+\sinh^2 t \sin^2\delta_{(2)}}\right) \quad \text{and} \tag{28}$$

$$\omega(k,\ell,t) = \left(\frac{i+\sinh t \sin\delta_{(1)}}{i-\sinh t \sin\delta_{(1)}}\right)^k \left(\frac{i+\cosh t \tan\delta_{(2)}}{i-\cosh t \tan\delta_{(2)}}\right)^\ell .$$

When $t = 0$ we recover a degenerate cK-net of a familiar form

$$f_0(k,\ell) = \begin{bmatrix} -2k\sin\delta_{(1)} \\ 0 \\ 0 \end{bmatrix} \quad \text{with} \quad n_0(k,\ell) = \begin{bmatrix} 0 \\ -\sin(2\ell\delta_{(2)}) \\ \cos(2\ell\delta_{(2)}) \end{bmatrix}. \tag{29}$$

The single Bäcklund transformation of the straight line can be solved in full generality in closed form; the derivation was performed using a mixture of hand and symbolic computation. Throughout this section we denote by subscript b the single Bäcklund transform of the straight line. For an initial choice $s_b(0,0) = e^{i\theta}, \theta \neq 0 \in (-\pi, \pi)$, the evolution recursion formulas (26) can be solved yielding

$$\begin{aligned} s_b(k,\ell) &= (-1)^\ell \left(-1 + \frac{2}{1 - ie^{\chi(k,\ell)}} \right), \quad \text{where} \\ \chi(k,\ell,\alpha) &= \log \tan \frac{\theta}{2} + k \log \frac{\sin\alpha + \sin\delta_{(1)}}{\sin\alpha - \sin\delta_{(1)}} - \ell \log \frac{\sin\left(\alpha - \delta_{(2)}\right)}{\sin\left(\alpha + \delta_{(2)}\right)}. \end{aligned} \tag{30}$$

Solving for the immersion and Gauß map explicitly using (27) we arrive at the following theorem.

Theorem 9. *The Bäcklund transformation of the straight line (f_0 and ω as in (28)) with parameter $\alpha \in (-\pi, \pi)$, together with its associated family with spectral parameter $\lambda = e^t \in \mathbb{R}$, is given by*

$$\begin{aligned} f_b(k,\ell,\alpha,t) &= f_0(k,\ell,t) + \frac{\sin\alpha}{\cosh t - \cos\alpha \sinh t} \begin{bmatrix} \tanh\chi \\ -\operatorname{sech}\chi \Re\omega \\ \operatorname{sech}\chi \Im\omega \end{bmatrix} \text{with} \\ n_b(k,\ell,\alpha,t) &= \frac{\sin\alpha}{\cosh t - \cos\alpha \sinh t} \begin{bmatrix} \operatorname{sech}\chi \\ \Im\hat{\omega} \\ \Re\hat{\omega} \end{bmatrix}, \end{aligned} \tag{31}$$

with χ defined as in (30) and for simplicity we set

$$\hat{\omega} = \omega(i \tanh\chi + (\cosh t \cot\alpha - \csc\alpha \sinh t)). \tag{32}$$

Remark 23. *In the most general case where the parameter line functions $\delta_{(1)}(k)$ and $\delta_{(2)}(\ell)$ vary, we have (up to reversing summation/product indices based on the signs of k, ℓ):*

$$\begin{aligned} \chi(k,\ell,\alpha) &= \log \tan \frac{\theta}{2} + \sum_{s=0}^{k-1} \log \frac{\sin\alpha + \sin\delta_{(1)}(s)}{\sin\alpha - \sin\delta_{(1)}(s)} - \sum_{s=0}^{\ell-1} \log \frac{\sin\left(\alpha - \delta_{(2)}(s)\right)}{\sin\left(\alpha + \delta_{(2)}(s)\right)}, \\ \omega(k,\ell,t) &= \prod_{s=0}^{k-1} \frac{i + \sinh t \sin\delta_{(1)}(s)}{i - \sinh t \sin\delta_{(1)}(s)} \prod_{s=0}^{\ell-1} \frac{i + \cosh t \tan\delta_{(2)}(s)}{i - \cosh t \tan\delta_{(2)}(s)}, \quad \text{and} \\ x(k,\ell,t) &= \sum_{s=0}^{k-1} \frac{\cosh t \sin\delta_{(1)}(s)}{1 + \sinh^2 t \sin^2\delta_{(1)}(s)} + \sum_{s=0}^{\ell-1} \frac{\sinh t \sin\delta_{(2)}(s) \cos\delta_{(2)}(s)}{1 + \sinh^2 t \sin^2\delta_{(2)}(s)}. \end{aligned} \tag{33}$$

We wish to highlight two special cases of the above theorem; the first provide a discrete analogue of Dini's surfaces in curvature line coordinates given by a family of Bäcklund transformations, while the latter provides a discrete analogue of Dini's surfaces in more general coordinates given by an associated family (see Figure 11).

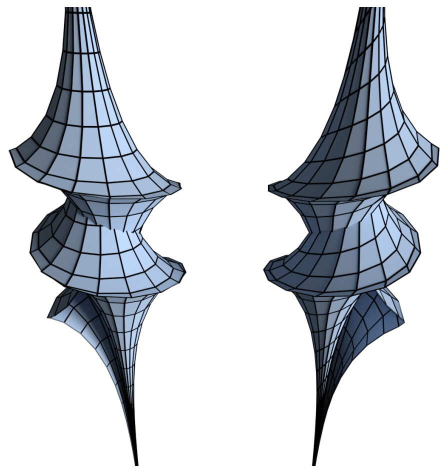

Figure 11: Two Dini nets. Note the subtle difference in how the cusp line aligns with the parameter polygons in the right version but not in the left.

Corollary 2. *Setting $t = 0$ in Theorem 9 yields the Bäcklund transformations of the straight line that are all cK-nets.*

$$f^{cK}_{dini}(k,\ell) = \begin{bmatrix} -2k\sin\delta_{(1)} \\ 0 \\ 0 \end{bmatrix} + \sin\alpha \begin{bmatrix} \tanh\chi \\ -\operatorname{sech}\chi\cos(2\ell\delta_{(2)}) \\ -\operatorname{sech}\chi\sin(2\ell\delta_{(2)}) \end{bmatrix} \quad \text{with}$$
$$n^{cK}_{dini}(k,\ell) = \begin{bmatrix} \operatorname{sech}\chi\sin\alpha \\ \tanh\chi\cos(2\ell\delta_{(2)})\sin\alpha - \cos\alpha\sin(2\ell\delta_{(2)}) \\ \tanh\chi\sin(2\ell\delta_{(2)})\sin\alpha + \cos\alpha\cos(2\ell\delta_{(2)}) \end{bmatrix}. \tag{34}$$

Corollary 3. *Setting $\alpha = -\frac{\pi}{2}$ in Theorem 9 yields the associated family of the Beltrami pseudosphere of revolution, see Figure 7.*

$$f_{pseudo}(k,\ell,t) = f_0(k,\ell,t) - \operatorname{sech} t \begin{bmatrix} \tanh(k\tau) \\ -\operatorname{sech}(k\tau)\Re\omega \\ \operatorname{sech}(k\tau)\Im\omega \end{bmatrix} \quad \text{with}$$
$$n_{pseudo}(k,\ell,t) = -\operatorname{sech} t \begin{bmatrix} \operatorname{sech}\chi \\ \tanh\chi\Re\omega + \sinh t\Im\omega \\ \tanh\chi\Im\omega - \sinh t\Re\omega \end{bmatrix}, \tag{35}$$

where $\tau = \log\frac{1-\sin\delta_{(1)}}{1+\sin\delta_{(1)}}$.

Remark 24. *It is clear that either setting $\alpha = -\frac{\pi}{2}$ in the cK-net family of Dini's surfaces or setting $t = 0$ in the associated family of the Pseudosphere we recover, after the change of variables $\epsilon = -2\sin\delta_{(1)}$ and $\phi = 2\delta_{(2)}$, the closed form of the Pseudosphere given in Remark 12.*

2.5.2 Special double transformations of the straight line

In this section we give closed form expressions for double Bäcklund transformations of the straight line with opposite (real or complex) parameters α and $-\alpha$, together with their associated families. Throughout we assume that $\alpha = \arcsin d$ for $d \geq 1$, so in particular, we can define μ by $e^{i\mu} = \tan\frac{\alpha}{2}$.

Such transformations are given by multiplying the straight line frame by a cK-net Lax matrix B that is the product of two rhombic K-net Lax matrices (5) with opposite parameters as discussed in Lemma 7. Its variables have the algebraic restrictions discussed in Theorem 5, i.e., unitary vertex variables s_0, s_{db} and an edge variable s_b that is only unitary for $e^{i\mu} = \pm 1$

$$B = \begin{pmatrix} s_b(e^{i\mu}s_{db} + \frac{1}{e^{i\mu}s_0}) & i(\lambda - \frac{s_{db}s_0}{\lambda}) \\ i(\lambda - \frac{1}{\lambda s_{db}s_0}) & \frac{1}{s_b}(\frac{e^{i\mu}}{s_{db}} + \frac{s_0}{e^{i\mu}}) \end{pmatrix}. \tag{36}$$

Given the Lax matrices for the straight line and the single Bäcklund transformation variable $s_b(k,\ell)$ (which becomes the edge quantity here), we can solve the recurrence relations (17) governing the evolution variable s_{db}. For simplicity we assume that $s_b(0,0) = i$, $s_{db}(0,0) = 1$, and that $\delta_{(1)}, \delta_{(2)}$ are constant. For $e^{i\mu} \neq \pm 1$ we find

$$\begin{aligned} s_0(k,\ell) &= (-1)^\ell, \\ s_b(k,\ell) &= (-1)^\ell \left(-1 + \frac{2}{1 - ie^{-(i\ell\kappa + k\tau)}}\right), \quad \text{and} \\ s_{db}(k,\ell) &= (-1)^\ell \left(-1 + \frac{2}{1 - i(\cot\mu \operatorname{sech}(k\tau)\sin(\ell\kappa))}\right), \end{aligned} \tag{37}$$

where $\kappa = 2\arctan(\sin\mu\tan\delta_{(2)})$ and $\tau = \log\frac{1-\sin\delta_{(1)}\cos\mu}{1+\sin\delta_{(1)}\cos\mu}$. Note that we performed a change of variables to split χ into a κ and τ part.

Theorem 10. *The immersion and Gauß map for the stationary breather with Bäcklund parameter $\mu \neq 0, \pi \in [0, 2\pi)$ and associated family parameter $\lambda = e^t$ of the straight line (with f_0, ω as in (28)) are given by*

$$f_{breather}(k,\ell,\mu,t) = f_0(k,\ell,t) + \\ 2A\begin{bmatrix} \cos\mu\sinh t \operatorname{sech}(k\tau)\sin(\ell\kappa)\cos(\ell\kappa) - \sin\mu\cosh t\sinh(k\tau) \\ \Im\omega\sin(\ell\kappa) - \Re\omega(\cos\mu\sinh t\tanh(k\tau)\sin(\ell\kappa) + \sin\mu\cosh t\cos(\ell\kappa)) \\ \Im\omega(\cos\mu\sinh t\tanh(k\tau)\sin(\ell\kappa) + \sin\mu\cosh t\cos(\ell\kappa)) + \Re\omega\sin(\ell\kappa) \end{bmatrix} \tag{38}$$

and

$$n_{breather}(k,\ell,\mu,t) = \\ \frac{A}{2\cosh(k\tau)}\begin{bmatrix} 4(\sin\mu\sinh t\cosh(k\tau)\cos(\ell\kappa) + \cos\mu\cosh t\sinh(k\tau)\sin(\ell\kappa)) \\ \Re\omega B - \Im\omega C \\ -\Re\omega C - \Im\omega B, \end{bmatrix} \tag{39}$$

where

$$
\begin{aligned}
A &= \frac{\sin(2\mu)\cosh(k\tau)}{(\cos(2\mu)+\cosh(2t))(\cos^2\mu\sin^2(\ell\kappa)+\cosh^2(k\tau)\sin^2\mu)},\\
B &= 2(\cos\mu\cosh t\sin(2\ell\kappa)-\sin\mu\sinh t\sinh(2k\tau)) \quad \text{and}\\
C &= \sin^2(\ell\kappa)(\sin(2\mu)+\cot\mu(\cosh(2t)+1))\\
&\quad -\cosh^2(k\tau)(\sin(2\mu)+\tan\mu(1-\cosh(2t))).
\end{aligned} \tag{40}
$$

Remark 25. *When $t = 0$ the resulting cK-net immersions agree with those found by Schief [?]. As he notes, for each rational number $0 < q < 1$, setting*

$$
\mu = -\arcsin(\cot\delta_{(2)}\tan(\delta_{(2)}q)) \tag{41}
$$

generates a surface that is closed in one lattice direction. The breather cK-net with $q = \frac{3}{5}$ is shown in Figure 10.

Naively setting $\mu = 0$ in the previous theorem does not yield Kuen's surface, however taking the limit as $\mu \to 0$ does. Alternatively, one could solve the recursion formulas for real $\tan\frac{\alpha}{2} = 1$. In the following theorem we have $\tau = \log\frac{1-\sin\delta_{(1)}}{1+\sin\delta_{(1)}}$.

Theorem 11. *Kuen's surface and its associated family $\lambda = e^t$, as a double transformation of the straight line, are given (with f_0, ω as in (28)) by*

$$
f_{Kuen}(k,\ell,t) = f_0(k,\ell,t) + \frac{2\cosh(k\tau)\operatorname{sech} t}{\cosh^2(k\tau)+4\ell^2\tan^2\delta_{(2)}}\begin{bmatrix} 2\ell\tan\delta_{(2)}\tanh t\operatorname{sech}(k\tau)-\sinh(k\tau)\\ 2\ell\tan\delta_{(2)}\Im\omega\operatorname{sech} t-\Re\omega\left(2\ell\tan\delta_{(2)}\tanh t\tanh(k\tau)+1\right)\\ \Im\omega\left(2\ell\tan\delta_{(2)}\tanh t\tanh(k\tau)+1\right)+2\ell\tan\delta_{(2)}\Re\omega\operatorname{sech} t \end{bmatrix} \tag{42}
$$

with Gauss map

$$
n_{Kuen}(k,\ell,t) = f_0(k,\ell,t) + \frac{\operatorname{sech}^2 t}{\cosh^2(k\tau)+4\ell^2\tan^2\delta_{(2)}}\begin{bmatrix} 2\left(2l\tan\delta_{(2)}\cosh t\sinh(k\tau)+\sinh t\cosh(k\tau)\right)\\ \Im\omega D+\Re\omega E\\ \Re\omega D-\Im\omega E, \end{bmatrix} \tag{43}
$$

where

$$
\begin{aligned}
D &= \left(1-\sinh^2 t\right)\cosh^2(k\tau)-4l^2\tan^2\delta_{(2)}\cosh^2 t \quad \text{and}\\
E &= 4l\tan\delta_{(2)}\cosh t-\sinh t\sinh(2k\tau).
\end{aligned} \tag{44}
$$

Remark 26. *When $t = 0$ we recover a cK-net Kuen's surface, as shown in Figure 9.*