Mathematisches Forschungsinstitut Oberwolfach

Report No. 34/2012

DOI: 10.4171/OWR/2012/34

Discrete Differential Geometry

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8th July – 14th July 2012

Mathematics Subject Classification (2000): n. n.

Abstracts

Convergence of Discrete Elastica

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(joint work with Sebastian Scholtes and Max Wardetzky)

The bending energy of a thin, naturally straight, homogeneous and isotropic elastic rod of length L is given by

$$F(\gamma) = \int_0^L |\kappa(s)|^2 \, \mathrm{d}s,$$

where $\gamma\colon [0, L] \to \mathbb{R}^m$ is the arclength parametrisation and $\kappa = \gamma''(s)$ the curvature vector. Consider the following boundary value problem: Given points P, $Q \in \mathbb{R}^m$ and unit vectors v, $w \in \mathbb{S}^{m-1}$ find the shapes of static elastic curves with clamped ends and fixed length. Defining the space

$$C = \left\{ \gamma \in L^2([0, L]; \mathbb{R}^m) \;\middle|\; \begin{array}{ll} \gamma' \in L^2([0, L]; \mathbb{S}^{m-1}), & \gamma(0) = P, \ \gamma(L) = Q, \\ \gamma'' \in L^2([0, L]; \mathbb{R}^m), & \gamma'(0) = v, \ \gamma'(L) = w \end{array} \right\},$$

this can be reformulated to find the minimizers of $F\colon C \to \mathbb{R}$.

A widely used discrete bending energy for a polygonal line $p = (p_0, p_1, \cdots, p_n)$ with $p_i \in \mathbb{R}^m$ is given by

$$F_n(p) = \sum_{i=1}^{n-1} \left(\frac{\varphi_i}{\ell_i}\right)^2 \ell_i,$$

where φ_i is the turning angle and ℓ_i is given by $\ell_i = \frac{1}{2}(|p_{i+1} - p_i| + |p_i - p_{i-1}|)$. We restrict ourselves to evenly segmented polygons, i. e. $|p_i - p_{i-1}| = \frac{L}{n}$ for all $i = 1, \ldots, n$. It is straightforward to formulate a discrete analogon of the boundary value problem above: Find the minimizers of $F_n : C_n \to \mathbb{R}$ with the discrete ansatz space

$$C_n = \left\{ (p_0, \ldots, p_n) \in (\mathbb{R}^m)^n \;\middle|\; \begin{array}{c} |p_i - p_{i-1}| = \frac{L}{n},\ p_0 = P,\ p_n = Q, \\ p_1 - p_0 = \frac{L}{n} v,\ p_n - p_{n-1} = \frac{L}{n} w \end{array} \right\}.$$

There has been an attempt by Bruckstein et al. [1] to relate $\operatorname{argmin}(F_n)$ and $\operatorname{argmin}(F)$ via techniques from the theory of epi-convergence. However, epi-convergence of F_n to F only guarantees that *some* minimizers of F can be approximated by those of F_n. We are able to improve this result in various ways:

- The metric on configuration space is strenghtened from Fréchet-distance to $W^{1,2}$-distance.
- We settle some subtleties concerning the length constraint.
- If certain growth conditions of F, F_n can be established, the method yields convergence rates for Hausdorff distance of $\operatorname{argmin}(F)$ and $\operatorname{argmin}(F_n)$.

As metric space we choose

$$X = \left\{ \gamma \in W^{1,\infty}([0, L]; \mathbb{R}^m) \mid \gamma' \in L^\infty([0, L]; \mathbb{S}^{m-1}),\ \gamma(0) = P,\ \gamma(L) = Q \right\}$$

with distance

$$d_X(\gamma_1, \gamma_2) = \left(\int_0^L d_{\mathbb{S}^{m-1}}(\gamma_1'(t), \gamma_2'(t))^2 \, \mathrm{d}t \right)^{\frac{1}{2}}, \qquad \gamma_1,\ \gamma_2 \in X.$$

Both C and C_n are contained in X and we extend F, F_n to X by

$$F(\gamma) = \begin{cases} F(\gamma), & \gamma \in C, \\ \infty, & \text{else}, \end{cases} \qquad\qquad F_n(\gamma) = \begin{cases} F_n(\gamma), & \gamma \in C_n, \\ \infty, & \text{else}. \end{cases}$$

In general, define

$$\operatorname{argmin}^\delta(F)_\varepsilon = \{x \in X \mid \exists y \in X : F(y) \le \inf(F) + \delta \text{ and } d_X(x, y) \le \varepsilon\}.$$

Our main result is

Theorem 1. *For given length and boundary data, there is $c > 0$ s. t.*

$$|\inf(F_n) - \inf(F)| \le \frac{c}{n},$$

$$\operatorname{argmin}(F_n) \subset \operatorname{argmin}^{\frac{c}{n}}(F)_{\frac{c}{n}} \quad \text{and} \quad \operatorname{argmin}(F) \subset \operatorname{argmin}^{\frac{c}{n}}(F_n)_{\frac{c}{n}}$$

hold.

If F and F_n grow quadratically at their respective minimizers (which appears to be the case generically, but we cannot prove this fact yet), this result implies Hausdorff convergence

$$\operatorname{argmin}(F_n) \overset{n\to\infty}{\longrightarrow} \operatorname{argmin}(F)$$

with convergence rate $\sqrt{\frac{1}{n}}$ in the metric space X.

The proof of Theorem 1 uses techniques which are very much related to the notions of epigraph distances and Attouch-Wets-convergence. (See for example Rockafellar and Wets [2], Chapter 7.) We translate these results to our situation and obtain the following sufficient conditions for Theorem 1 to hold:

- For every global minimizer $\gamma \in C$ of F there is $p \in C_n$ with
$$d_X(p, \gamma) \leq \frac{c}{n} \quad \text{and} \quad F_n(p) \leq F(\gamma) + \frac{c}{n}.$$
- For every global minimizer $p \in C_n$ of F_n there is $\gamma \in C$ with
$$d_X(\gamma, p) \leq \frac{c}{n} \quad \text{and} \quad F(\gamma) \leq F_n(p) + \frac{c}{n}.$$

Finally, we show that these conditions are actually fulfilled. Two things are crucial: (i) Minimizers of F have higher regularity than $W^{2,2}$, in particular κ' is bounded. (ii) The energy F and $\|\kappa'\|_{L^\infty}$ of a curve give $\frac{c}{n}$-bounds for the error of suitably chosen polygonal approximations.

References

[1] A. M. Bruckstein, A. N. Netravali and T. J. Richardson, *Epi-convergence of discrete elastica*, Appl. Anal. **79** (2001), 137–171.

[2] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, Grundlehren der mathematischen Wissenschaften, Springer (2004).

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